EFFECT OF PÉCLET NUMBER ON THE CONVERGENCE OF INTEGRAL-TRANSFORM AND FINITE-VOLUMES SOLUTIONS IN THERMALLY DEVELOPING FLOW

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Abstract. *The current work provides a comparison between two different methodologies for solving convection-diffusion problems: the Generalized Integral Transform Technique (GITT) and the Finite Volumes Method (FVM). The problem of thermally developing laminar flow between parallel plates is selected for illustrating purposes. Different solution strategies can be employed for both methods; consequently, several different comparisons can be performed. This study focuses on evaluating the effect of varying the Péclet number (based on a transversal length) on the convergence of both methodologies for thermal developing flow. Hydrodynamic development is considered, such that a Hagen-Poiseuile velocity profile is used; in addition, the results of a simplified slug-flow situation are also presented. Once comparisons are performed, advantages and disadvantages of each methodology are discussed. The results indicate that, in general, the Integral Transform Technique presents a better convergence rate.*

Keywords: *Integral Transform, Finite Volumes, Forced Convection, Laminar Duct Flow, Parallel Plates Channels*

NOMENCLATURE

D_H	hydraulic diameter
H	distance between plates
I	number of volumes in the axial direction
J	number of volumes in the transversal direction
L	characteristic dimension in x-direction
N	norm of eigenfunctions
Nu_H	Nusselt Number
Pe_H	Péclet Number
T_s	surface temperature
T_0	entrance temperature
T_m	bulk or average mixing temperature
u	velocity component in x-direction
$\bar{u}$	average velocity in x-direction
x, y	cartesian coordinates
Y_n	eigenfunctions

Greek Symbols

α	thermal diffusivity
ξ, η	dimensionless coordinates
ϕ	arbitrary function
λ_n	eigenvalues
μ	dynamic viscosity
θ	dimensionless temperature
$\bar{\theta}_n$	transformed dimensionless temperature
$\hat{\theta}_i$	discretized dimensionless temperature
$\xi_{\max}$	dimensionless channel length

Superscripts and overscripts

$*$	dimensionless value
$\hat{}$	associated to the FVM solution

1. INTRODUCTION

Numerical methods based on domain discretization have been employed for the solution of convection-diffusion problems for about half century. On a smaller time scale meshless techniques have been gradually emerging as competitive alternatives to traditional discretization-based methods. One such approach is the so called Generalized Integral Transform Technique (GITT) (Cotta, 1993), which has been successfully applied to a variety of convection-diffusion problems. In the realm of discrete methods, the Finite Volume Method (FVM) (Patankar, 1980) appears as widely used option to a variety of convection-diffusion problems, due to its conservative nature and ease of application. Nevertheless, as with any discrete method, approximations to integrals derivatives in terms of nodal points on a computational domain are necessary. This results in a solution error, which gradually decays with grid refinement. Integral Transform solutions are sought by expanding the unknown potentials in terms of infinite series of orthogonal functions that arise from eigenvalue problems. Naturally, a truncation error is introduced since the infinite series representation must be made finite for computational implementation. Then again, this error decreases as the number of terms are increased and the solution converges to a final value. Due to the nature of the series representation, error estimates can be easily obtained from this method, which results in a better control of the global solution error. The usual drawback associated with this approach is the elaborate mathematical manipulation; however, this effort can be considerably minimized with the employment of symbolical computation (Wolfram, 2003). Because of the inherent characteristics between the two type of approaches mentioned above, one can expect that a different numerical behavior will be seen for the GITT and FVM when different kind of problems are considered.

For convective heat transfer in duct flow, different investigations were carried out employing integral transforms. Among recent advancements, one should mention (Macêdo, Maneschy et al., 2000; Nacimento, Quaresma et al., 2006, 2002), which deals with non-Newtonian flows in circular-shaped ducts, (Maia, Aparecido et al., 2006), which presents a solution for non-Newtonian flows in elliptical cross-section ducts, and (Lima, Quaresma et al., 2007), which investigates the MHD flow and heat transfer within parallel-plates channels. For flow in ducts of arbitrary geometry, some particular solutions have been developed (Aparecido and Cotta, 1990; Barbuto and Cotta, 1997; Ding and Manglik, 1996; Guerrero, Quaresma et al., 2000); nonetheless, a general methodology was described in (Sphaier and Cotta, 2000, 2002), being potentially promising for these types of geometries.

Although there are several studies that separately deal with FVM or GITT solutions to convection-diffusion problems, there is a relative lack of comparative studies. A recent investigation compared the performance of GITT and FVM solutions for steady thermally developing laminar channel flow (Chalhub, Dias et al., 2008). however, only results of the simplified cases with large Péclet values were examined. This paper extends these comparison to a broader number of cases, by examining the numerical performance of GITT and FVM if the Péclet number is allowed to assume smaller values. Numerical results are calculated using the *Mathematica* system.

2. MATHEMATICAL FORMULATION

2.1 Problem presentation

The studied problem is that of heat transfer in steady incompressible laminar flow between two parallel plates. The flow is considered hydrodynamically developed, but thermally developing. The problem is given by the following dimensionless equations:

$$\frac{1}{2}\,u^*\,\frac{\partial\theta}{\partial\xi} = \mathrm{Pe}_H^{-2}\frac{\partial^2\theta}{\partial\xi^2} + \frac{\partial^2\theta}{\partial\eta^2}, \qquad \text{for} \qquad 0 \le \xi < \infty \qquad \text{and} \qquad 0 \le \eta \le 1, \tag{1}$$

$$\theta(\xi,1) = 0, \qquad \left(\frac{\partial\theta}{\partial\eta}\right)_{\eta=0} = 0, \qquad \theta(0,\eta) = 1, \qquad \left(\frac{\partial\theta}{\partial\xi}\right)_{\xi\to\infty} = 0, \tag{2}$$

where the dimensionless quantities are given by

$$\theta = \frac{T-T_0}{T_{in}-T_0}, \qquad \eta = \frac{y}{H/2}, \qquad \xi = \frac{x}{L}, \tag{3}$$

and the value of L is chosen from a scale analysis of the thermal entry length:

$$L = \frac{H}{2}\,\mathrm{Pe}_H, \qquad \text{with} \qquad \mathrm{Pe}_H = \frac{\bar{u}\,H}{\alpha}. \tag{4}$$

The dimensionless velocity is given by the Hagen-Poiseuille profile:

$$u^* = \frac{u}{\bar{u}} = \frac{3}{2}\,(1-\eta^2). \tag{5}$$

However, a if simplified slug-flow case is considered, $u^* = 1$ and the previous equations are modified.

The Nusselt number in terms of the dimensionless variables is given by:

$$\mathrm{Nu}_{D_H} = \frac{-4\,(\partial\theta/\partial\eta)_{\eta=1}}{\int_0^1 u^*\theta\,\mathrm{d}\eta}. \tag{6}$$

2.2 Finite Volumes Method

The solution of the studied problem via finite volumes is accomplished by integrating eq. (1) within a finite volume of height $\Delta\eta = 1/J$ and employing second-order approximations for integration and interpolation, which leads to the following discretized system:

$$-\mathrm{Pe}_H^{-2}\,\frac{\mathrm{d}^2\hat{\theta}_j}{\mathrm{d}\xi^2} + \frac{1}{2}\,\hat{u}_j^*\,\frac{\mathrm{d}\hat{\theta}_j}{\mathrm{d}\xi} = F_j(\xi), \qquad \hat{\theta}_j(0) = 1, \qquad \left(\frac{\mathrm{d}\hat{\theta}_j}{\mathrm{d}\xi}\right)_{\xi=\xi_{\max}} = 0, \tag{7}$$

for $j = 1, 2, \ldots, J$. The F-functions, which carry all the η-discretization information, are given by:

$$F_j(\xi) = \frac{\hat{\theta}_{j+1} - \hat{\theta}_j}{\Delta\eta^2}, \qquad \text{for} \qquad j = 1, \tag{8}$$

$$F_j(\xi) = \frac{\hat{\theta}_{j+1} - 2\,\hat{\theta}_j + \hat{\theta}_{j-1}}{\Delta\eta^2}, \qquad \text{for} \qquad 1 < j < J, \tag{9}$$

$$F_j(\xi) = \frac{\hat{\theta}_{j-1} - 3\,\hat{\theta}_j}{\Delta\eta^2}, \qquad \text{for} \qquad j = J. \tag{10}$$

For cases with small to moderate Péclet numbers, this system is solved numerically using the **NDSolve** function available in the *Mathematica* software. Simpified solutions for slug flow are also obtained by setting $u^* = 1$. Using the obtained solutions, the Nusselt number is then calculated from eq. (6), by numerically computing the derivative and integral.

2.2.1 Multidimensional discretized solutions

Although the previous solutions – involving explicit discretization in a single spatial variable – are simple to implement, there are some numerical problems. When the large Péclet approximation is considered, the resulting system comprises an initial-value problem, which is easily handled by the ODE solver (**NDSolve**); however, if other Péclet values are considered, the axial diffusion terms must be maintained, and a coupled boundary-value system needs to be solved. Aside from small Péclet values, this system is very stiff and its numerical integration (as previously described) becomes unfeasible. Nevertheless, for these cases a solution involving FVM discretization in both variables can be employed. Considering that I and J are, respectively, is the number of volumes in the ξ and η directions, and using centered, second order approximations, the resulting discretized system is written in the following form:

$$\hat{\boldsymbol{M}}\,\hat{\boldsymbol{\theta}} + \hat{\boldsymbol{b}} = \boldsymbol{0}, \tag{11}$$

where the coefficients of $\hat{\boldsymbol{M}}$ and $\hat{\boldsymbol{b}}$ are given by:

- for $k = 1$:

$$\hat{M}_{k,k} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} - \frac{3}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{1}{\Delta\eta^2}, \qquad \hat{M}_{k,k+1} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k+I} = \frac{1}{\Delta\eta^2}, \tag{12}$$

$$\hat{b}_k = \frac{\hat{u}_j^*}{2\,\Delta\xi} + \frac{2}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{13}$$

- for $1 < k < I$:

$$\hat{M}_{k,k} = -\frac{2}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{1}{\Delta\eta^2}, \qquad \hat{M}_{k,k+1} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{14}$$

$$\hat{M}_{k,k-1} = \frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k+I} = \frac{1}{\Delta\eta^2}, \qquad \hat{b}_k = 0, \tag{15}$$

- for $k = I$:

$$\hat{M}_{k,k} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} - \frac{1}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{1}{\Delta\eta^2}, \qquad \hat{M}_{k,k+1} = 0, \tag{16}$$

$$\hat{M}_{k,k-1} = \frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k+I} = \frac{1}{\Delta\eta^2}, \qquad \hat{b}_k = 0, \tag{17}$$

- for $k = 1 + j\,I$ and $1 < j < J$:

$$\hat{M}_{k,k} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} - \frac{3}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{2}{\Delta\eta^2}, \qquad \hat{M}_{k,k-1} = 0, \tag{18}$$

$$\hat{M}_{k,k+1} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k-I} = \frac{1}{\Delta\eta^2}, \qquad \hat{M}_{k,k+I} = \frac{1}{\Delta\eta^2}, \tag{19}$$

$$\hat{b}_k = \frac{\hat{u}_j^*}{2\,\Delta\xi} + \frac{2}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{20}$$

- for $k = (j+1)\,I$ and $1 < j < J$:

$$\hat{M}_{k,k} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} - \frac{1}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{2}{\Delta\eta^2}, \qquad \hat{M}_{k,k-1} = \frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{21}$$

$$\hat{M}_{k,k+1} = 0, \qquad \hat{M}_{k,k-I} = \frac{1}{\Delta\eta^2}, \qquad \hat{M}_{k,k+I} = \frac{1}{\Delta\eta^2}, \qquad \hat{b}_k = 0, \tag{22}$$

- for $k = (J-1)\,I + 1$:

$$\hat{M}_{k,k} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} - \frac{3}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{3}{\Delta\eta^2}, \qquad \hat{M}_{k,k-1} = 0, \tag{23}$$

$$\hat{M}_{k,k+1} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k-I} = \frac{1}{\Delta\eta^2}, \qquad \hat{b}_k = \frac{\hat{u}_j^*}{2\,\Delta\xi} + \frac{2}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{24}$$

- for $k = (J-1)\,I + i$ and $1 < I < I$:

$$\hat{M}_{k,k} = -\frac{2}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{3}{\Delta\eta^2}, \qquad \hat{M}_{k,k-1} = \frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{25}$$

$$\hat{M}_{k,k+1} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k-I} = \frac{1}{\Delta\eta^2}, \qquad \hat{b}_k = 0, \tag{26}$$

- for $k = I\,J$:

$$\hat{M}_{k,k} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} - \frac{1}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{3}{\Delta\eta^2}, \qquad \hat{M}_{k,k-1} = \frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k-I} = \frac{1}{\Delta\eta^2}, \tag{27}$$

$$\hat{b}_k = 0, \tag{28}$$

- for all other $k = i + j\,I$ combination:

$$\hat{M}_{k,k} = -\frac{2}{\Delta\xi^2 \mathrm{Pe}_H^2} - \frac{2}{\Delta\eta^2}, \qquad \hat{M}_{k,k-1} = \frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \tag{29}$$

$$\hat{M}_{k,k+1} = -\frac{\hat{u}_j^*}{4\,\Delta\xi} + \frac{1}{\Delta\xi^2\,\mathrm{Pe}_H^2}, \qquad \hat{M}_{k,k-I} = \frac{1}{\Delta\eta^2}, \qquad \hat{M}_{k,k+I} = \frac{1}{\Delta\eta^2}, \qquad \hat{b}_k = 0, \tag{30}$$

where $\Delta\xi = \xi_{\max}/I$ and the remaining $\hat{M}_{k,l}$ coefficients are zero. The solution to this system is performed by defining $\hat{\boldsymbol{M}}$ as a sparse array and using the *Mathematica* **LinearSolve** function. The Nusselt number is calculated using numerical integration and differentiation.

2.3 Generalized Integral Transform Technique

The Integral Transform solution of the considered problem is accomplished employing the Generalized Integral Transform Technique (Cotta, 1993). The solution of the problem is started by defining the transformation pair

$$\text{Transform} \quad \Longrightarrow \quad \bar{\theta}_n(\xi) = \int_0^1 \theta(\xi,\,\eta) Y_n(\eta)\,\mathrm{d}\eta, \tag{31}$$

$$\text{Inversion} \quad \Longrightarrow \quad \theta(\xi,\,\eta) = \sum_{n=1}^{\infty} \frac{\bar{\theta}_n(\xi)\,Y_n(\eta)}{N(\lambda_n)}, \tag{32}$$

where Y_n's are orthogonal solutions to a Sturm-Liouville eigenvalue problem. For the convection-diffusion problem considered in this work, the following eigenvalue problem is selected:

$$Y_n''(\eta) + \lambda_n^2 Y_n(\eta) = 0, \qquad \text{for} \qquad 0 \le \eta \le 1, \qquad Y'(0) = 0, \qquad Y(1) = 0. \tag{33}$$

The previous problem leads to infinite nontrivial solutions in the form:

$$Y_n(\eta) = \cos(\lambda_n\,\eta), \qquad \text{with} \qquad \lambda_n = \left(n - \frac{1}{2}\right)\pi, \qquad \text{for} \qquad n = 1, 2, 3, \ldots \tag{34}$$

The norm of the Y_n eigenfunctions are given by:

$$N(\lambda_n) \;=\; \int_0^1 Y_n^2(\eta)\,\mathrm{d}\eta \;=\; \frac{1}{2}. \tag{35}$$

The transformation of the given problem is accomplished by multiplying eq. (1) by Y_n, integrating within $0 \le \eta \le 1$, and applying the inversion formula (32) to the non-transformable terms. This process yields

$$\mathrm{Pe}_H^{-2}\,\bar{\theta}_n''(\xi) \;-\; \frac{1}{2}\sum_{m=1}^{\infty} A_{n,m}\,\bar{\theta}_m'(\xi) \;-\; \lambda_n^2\bar{\theta}_n(\xi) \;=\; 0, \tag{36}$$

with the boundary conditions

$$\bar{\theta}_n(0) \;=\; b_n \;=\; \int_0^1 Y_n(\eta)\,\mathrm{d}\eta \qquad \text{and} \qquad \lim_{\xi\to\infty}\bar{\theta}_n'(\xi) \;=\; 0, \tag{37}$$

where the $A_{n,m}$ coefficients are given by:

$$A_{n,m} = \frac{1}{N(\lambda_m)}\int_0^1 u^*(\eta)\,Y_n(\eta)\,Y_m(\eta)\,\mathrm{d}\eta. \tag{38}$$

For a general case of small to moderate Péclet numbers with Hagen-Poiseiulle flow, this boundary value problem is solved numerically using the *Mathematica* function **NDSolve** and the dimensionless temperature is calculated using the inversion formula (32). For simpler cases, as described below, fully analytical solutions can be obtained. Regardless of the simplification considered, the Nusselt number is computed from the following expression:

$$\mathrm{Nu}_{D_H} = \frac{-4\sum\limits_{n=1}^{\infty}\bar{\theta}_n/N(\lambda_n)\,Y_n'(1)}{\sum\limits_{n=1}^{\infty}\bar{\theta}_n/N(\lambda_n)\,\int_0^1 u^* Y_n \mathrm{d}\eta}. \tag{39}$$

2.3.1 Slug-flow

If slug flow is considered, the ODE system (36) is decoupled, since $A_{n,m} = \delta_{n,m}$, thereby resulting in the following equation for the transformed potentials:

$$\mathrm{Pe}_H^{-2}\,\bar{\theta}_n''(\xi) \;-\; \frac{1}{2}\,\bar{\theta}_n'(\xi) \;-\; \lambda_n^2\bar{\theta}_n(\xi) \;=\; 0, \tag{40}$$

which yields

$$\bar{\theta}_n(\xi) \;=\; b_n\,\exp\left(\frac{\mathrm{Pe}_H^2\,\xi}{4}\right)\,\frac{4\,\beta_n\cosh(\beta_n(\xi_{\max}-\xi)) \;+\; \mathrm{Pe}_H^2\sinh(\beta_n(\xi_{\max}-\xi))}{4\,\beta_n\cosh(\beta_n\,\xi_{\max}) \;+\; \mathrm{Pe}_H^2\sinh(\beta_n\,\xi_{\max})}, \tag{41}$$

where the β_n coefficients are given by

$$\beta_n \;=\; \frac{\mathrm{Pe}_H}{4}\,\sqrt{\mathrm{Pe}_H^2 + 16\,\lambda_n^2}, \tag{42}$$

and the temperature profile is obtained using the inversion formula (32).

2.3.2 Hagen-Poiseuille flow: analytical integration

Equations (36) and (37) can be written in the following matrix form:

$$\bar{\boldsymbol{\theta}}''(\xi) \;-\; \boldsymbol{B}\,\bar{\boldsymbol{\theta}}'(\xi) \;-\; \boldsymbol{D}\,\bar{\boldsymbol{\theta}}(\xi) \;=\; 0, \qquad \bar{\boldsymbol{\theta}}(0) \;=\; \boldsymbol{b}, \qquad \bar{\boldsymbol{\theta}}'(\xi_{\max}) \;=\; \boldsymbol{0}, \tag{43}$$

in which the coefficients of $\boldsymbol{b}$ are given by eq. (37) and matrices $\boldsymbol{B}$ and $\boldsymbol{D}$ are given by

$$B_{n,m} \;=\; \frac{1}{2}\,\mathrm{Pe}_H^2\,A_{n,m}, \qquad D_{n,n} \;=\; \mathrm{Pe}_H^2\,\lambda_n^2\,\delta_{n,m}, \tag{44}$$

where $\delta_{n,m}$ is the Kronecker delta. This system can be converted to a first order initial-value problem if the boundary condition at $\xi_{\max}$ is replaced by an initial condition and a new variable is introduced:

$$\bar{\boldsymbol{\theta}}'(0) \;=\; \boldsymbol{p}, \qquad \bar{\boldsymbol{\theta}}'(\xi) \;=\; \bar{\boldsymbol{\phi}}(\xi), \tag{45}$$

yielding

$$\frac{\mathrm{d}}{\mathrm{d}\xi}\begin{Bmatrix}\bar{\boldsymbol{\phi}}\\ \bar{\boldsymbol{\theta}}\end{Bmatrix} = \left(\begin{array}{c|c}\boldsymbol{B} & \boldsymbol{D}\\ \hline \boldsymbol{I} & \boldsymbol{0}\end{array}\right)\begin{Bmatrix}\bar{\boldsymbol{\phi}}\\ \bar{\boldsymbol{\theta}}\end{Bmatrix}, \tag{46}$$

where $\boldsymbol{I}$ is the identity matrix, and $\boldsymbol{0}$ is a zero matrix. With this consideration, an analytical solution to the transformed potentials can be obtained in terms of a matrix exponential:

$$\begin{Bmatrix}\bar{\boldsymbol{\phi}}\\ \bar{\boldsymbol{\theta}}\end{Bmatrix} = \boldsymbol{C}\begin{Bmatrix}\boldsymbol{p}\\ \boldsymbol{b}\end{Bmatrix}, \qquad \text{with} \qquad \boldsymbol{C} = \exp\left(\left(\begin{array}{c|c}\boldsymbol{B} & \boldsymbol{D}\\ \hline \boldsymbol{I} & \boldsymbol{0}\end{array}\right)\xi\right). \tag{47}$$

With the previous analytical form, a shooting scheme using a Newton-Raphson routine (performed by the *Mathematica* **FindRoot** function) is used to iteratively calculate the appropriate value of $\boldsymbol{p}$ that satisfies the boundary condition at $\xi = \xi_{\max}$, given by eq. (43).

3. RESULTS AND DISCUSSION

Following the previous sections, the Nusselt number is calculated for four different positions ($\xi = 0.001$, 0.01, 0.1 and 1) and different values of the Péclet number, using both methodologies. Results for two types of flow are presented: slug flow and Hagen-Poiseuille flow. Table 1 shows Nusselt values obtained with the Integral Transform Method for slug flow with $\mathrm{Pe}_H = 1$ and $\mathrm{Pe}_H = 10$. As can be seen, the convergence rate is much worse for positions near the channel entrance (smaller values of ξ). Also, as Péclet is decreased the convergence rate is also diminished.

Table 1. Nusselt numbers for slug-flow (GITT).

	$\mathrm{Pe}_H = 10$					$\mathrm{Pe}_H = 1$			
$n_{\max}$	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$	$n_{\max}$	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
5	39.7501	27.5023	10.7213	9.86960	5	41.4542	39.4699	25.8967	10.3237
10	72.6080	33.4499	10.7213	9.86960	10	80.6022	71.9542	31.1657	10.3237
20	124.937	34.9693	10.7213	9.86960	20	157.153	123.655	32.5019	10.3237
40	191.300	35.0384	10.7213	9.86960	30	231.370	161.511	32.5598	10.3237
80	245.711	35.0385	10.7213	9.86960	40	303.309	189.197	32.5623	10.3237
100	255.819	35.0385	10.7213	9.86960	50	373.033	209.435	32.5625	10.3237
120	261.213	35.0385	10.7213	9.86960	100	690.706	252.906	32.5625	10.3237
140	264.091	35.0385	10.7213	9.86960	200	1194.41	263.830	32.5625	10.3237
160	265.627	35.0385	10.7213	9.86960	300	1562.40	264.302	32.5625	10.3237
180	266.446	35.0385	10.7213	9.86960	400	1831.21	264.322	32.5625	10.3237
200	266.883	35.0385	10.7213	9.86960	500	2027.57	264.323	32.5625	10.3237
250	267.279	35.0385	10.7213	9.86960	1000	2449.00	264.323	32.5625	10.3237
300	267.361	35.0385	10.7213	9.86960	2000	2554.83	264.323	32.5625	10.3237
350	267.378	35.0385	10.7213	9.86960	3000	2559.40	264.323	32.5625	10.3237
400	267.382	35.0385	10.7213	9.86960	4000	2559.60	264.323	32.5625	10.3237
500	267.383	35.0385	10.7213	9.86960	5000	2559.61	264.323	32.5625	10.3237

Next, table 2 shows the Nusselt values calculated with the GITT for Hagen-Poiseuille flow, for the same values of Péclect and axial positions. As observed, a similar behavior occurs, with the convergence rate being better for positions far from the inlet and for lager values of Péclet. Comparing the results for the two types of flow, one notes that the convergence is better for $\xi = 0.1$ and $\xi = 1$ for slug-flow, in which 5 terms are sufficient for obtaining a six-digit converged solution for $\xi = 1$ ($\mathrm{Pe}_H = 1$ and $\mathrm{Pe}_H = 10$) and $\xi = 0.1$ ($\mathrm{Pe}_H = 10$). For Hagen-Poiseuille flow, at least 20 terms are necessary for obtaining the same precision at these positions. Nevertheless, for positions closer to the channel entrance, the superior convergence seen in the slug flow case cannot be observed.

The following tables present the results calculated using the Finite Volumes Method for a variety of grids. Table 3 displays the results for slug flow while table 4 shows the results for Hagen-Poiseuille flow. Since for slug flow the Integral Transform methodology provided a closed form, simple, analytical solution a fully converged solution was calculated and included as the exact result for comparisons. As one can observe, similar trends seen with the Integral Transform solution are repeated here. The convergence rate is better for large Péclet values and for positions away from the channel entrance. For those cases, six converged digits were obtained for the more refined grids. Nevertheless, for most cases a much lower number of converged digits were obtained for any of the presented grids. Comparing the results of the different methodologies by examining the number of equations necessary for obtaining the same precision, it is seen that the Integral Transform solution yields a much superior convergence rate when compared to the finite volumes one. This result is in agreement with the observations done in (Chalhub, Dias et al., 2008), for a simpler version of the problem.

Table 2. Nusselt numbers for Hagen-Poiseuille flow (GITT).

$\mathrm{Pe}_H = 10$						$\mathrm{Pe}_H = 1$					
$n_{\max}$	WP	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$	$n_{\max}$	WP	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
5	100	37.6242	23.5100	8.15700	7.74044	5	100	39.7508	37.5555	23.1993	8.45041
10	200	69.5288	27.7774	8.14983	7.73986	10	100	78.8666	69.6035	27.9173	8.45012
20	300	120.003	28.7805	8.14897	7.73982	20	100	155.286	120.397	29.0950	8.45011
30	500	156.839	28.8167	8.14889	7.73982	30	100	229.341	157.494	29.1452	8.45010
40	600	183.733	28.8168	8.14887	7.73982	40	100	301.107	184.589	29.1472	8.45010
50	700	203.372	28.8164	8.14886	—	50	100	370.653	204.379	29.1472	8.45010
60	800	217.714	28.8162	8.14886	—	60	100	438.048	218.833	29.1472	8.45010
70	1000	228.188	28.8161	—	—	70	100	503.358	229.390	29.1472	8.45010
80	1100	235.837	28.8160	—	—	80	200	566.649	237.101	29.1472	8.45010
90	1300	241.424	28.8160	—	—	90	200	627.982	242.733	29.1472	8.45010
100	1400	245.504	28.8160	—	—	100	200	687.418	246.847	29.1472	8.45010
110	1500	248.483	—	—	—	110	200	745.016	249.851	29.1472	8.45010
120	1700	250.660	—	—	—	120	200	800.833	252.046	—	—
130	1800	252.249	—	—	—	130	200	854.923	253.649	—	—
140	1900	253.410	—	—	—	140	200	907.341	254.820	—	—
150	2100	254.258	—	—	—	150	300	958.137	255.675	—	—
160	2200	254.877	—	—	—	200	300	1189.50	257.510	—	—
170	2300	255.329	—	—	—	250	400	1387.24	257.891	—	—
180	2500	255.659	—	—	—	300	500	1556.23	257.970	—	—

It should be mentioned that the solution strategy of discretizing in the η-direction and solving the resulting coupled system (7) using an ODE solver was unsuccessful. With this strategy, the ODE solver could not handle grids with over 50 divisions. Hence, the discretization in both directions became necessary. The same problem was seen with the GITT solution, if system (43) was tried to be solved numerically using and ODE solver. However, in that case, a matrix exponential analytical solution combined with a numerical shooting routine was used to avoid numerical integration. This idea was also tried for the FVM solutions. However, due to the much higher number of equations required by the FVM, evaluating matrix exponentials became unfeasible, such that the only possible solution was the two-dimensional discretization used in system (11). The stiffness of systems (7) and (43) is reduced for smaller Péclet numbers; however, the convergence of those cases is worse, requiring a greater number of terms (GITT) and more grid divisions (FVM) for obtaining the same precision seen for higher Péclet values.

In table 2, besides presenting the convergence evolution with the truncation order ($n_{\max}$), the required working precision (WP) for evaluating the matrix exponential in the Integral Transform solution is also shown. This quantity consists of the number of decimal digits needed for the calculations. As seen, this value is clearly higher for larger Péclet values, due to the increased stiffness of the transformed system. In addition WP increases with the truncation order, such that for $n_{\max}$ lager than 100 a significant computational effort is required, especially for high Péclet values. Although this may seem as a problem solely associated with the GITT solution, the same strategy was attempted with the FVM; nevertheless, due a much higher required number of equations, this approach becomes inviable for the FVM.

4. SUMMARY AND CONCLUSIONS

The solution for thermal developing flow in a parallel-plates channel was carried out using two very different methodologies: the Finite Volumes Method and the Generalized Integral Transform Technique. Initially, both solutions were aimed at transforming the transversal direction, either by discretization (FVM) or integral transformation (GITT), resulting in linear coupled ODE systems. Due to the boundary conditions involved of theses systems, its numerical integration was only feasible to a limited number of equations, allowing only coarse grids (FVM) and low truncation orders (GITT) to be used. For the simple slug flow situation, the Integral Transform solution resulted in a decoupled ODE system, which allowed a simple analytical solution to be obtained. For other cases, an alternative route was sought. The coupled GITT system was handled using an analytical matrix exponential solution to an associated initial value problem, and the unknown additional initial (inlet) conditions to this problem were calculated using a numerical shooting scheme. This strategy was shown to be feasible for systems not much larger than 150 equations, especially for larger Péclet values. For the GITT solution this limit still allowed yielded reasonable convergence rates; nevertheless, for the FVM solution, the elevated number of equations required for obtaining a similar precision made this strategy inapplicable to this method. Because of this, the discretization in the axial direction was also required. A comparison of the results obtained with both methods showed that, in general, better convergence rates are seen for positions upstream (away from the channel entrance) and for higher Péclet values. Analyzing the number of equations needed for obtaining the same precision, it was seen that the FVM requires a greater amount for obtaining the same results.

This work extends the analysis performed in (Chalhub, Dias et al., 2008) for a wider number of cases. The results are

Table 3. Nusselt numbers for slug-flow (FVM).

$\mathrm{Pe}_H = 10$						$\mathrm{Pe}_H = 1$					
I	J	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$	I	J	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
12	12	144.614	113.466	-23.2099	9.83306	12	12	145.868	126.595	35.106	10.3237
12	25	299.720	229.898	-70.0798	9.85179	12	25	300.286	238.568	6.00708	10.3083
12	50	598.030	454.055	-159.489	9.86067	12	50	596.65	448.451	-52.9306	10.3053
12	100	1194.67	902.511	-337.928	9.86513	12	100	1189.43	868.72	-167.625	10.3046
12	200	2387.08	1799.00	-694.682	9.86960	12	200	2375.14	1710.78	-394.107	10.3045
12	400	4773.64	3593.01	-1407.94	9.86960	12	400	4746.74	3396.01	-845.314	10.3044
25	12	140.616	79.3671	11.0194	9.85949	25	12	145.627	125.128	42.9731	10.3399
25	25	289.916	146.083	10.9712	9.86455	25	25	297.025	213.950	36.6179	10.3238
25	50	577.146	275.382	10.9624	9.86706	25	50	582.802	341.543	33.4288	10.3205
25	100	1151.69	534.677	10.9603	9.86834	25	100	1151.79	576.749	32.7729	10.3197
25	200	2300.00	1053.50	10.9598	9.86960	25	200	2289.91	1048.73	32.6544	10.3196
25	400	4598.28	2091.69	10.9597	9.86960	25	400	4566.93	1998.52	32.6289	10.3195
50	12	134.770	44.5090	10.8198	9.86677	50	12	145.715	126.447	43.3002	10.3433
50	25	272.773	42.8670	10.7881	9.86806	50	25	296.023	211.825	36.3833	10.3270
50	50	538.332	41.0815	10.7817	9.86882	50	50	570.839	282.865	33.1725	10.3237
50	100	1069.81	39.9545	10.7802	9.86924	50	100	1100.19	305.040	32.6639	10.3229
50	200	2132.99	39.3385	10.7798	9.86960	50	200	2148.96	295.649	32.5736	10.3227
50	400	4258.85	39.0187	10.7797	9.86960	50	400	4247.24	282.969	32.5527	10.3227
100	12	128.980	37.5839	10.7749	9.86909	100	12	145.832	127.653	43.4034	10.3442
100	25	246.909	1.62527	10.7453	9.86918	100	25	296.423	218.791	36.3752	10.3278
100	50	470.410	-72.1648	10.7392	9.86938	100	50	567.219	299.010	33.1692	10.3245
100	100	918.199	-216.200	10.7378	9.86949	100	100	1055.20	274.021	32.6809	10.3237
100	200	1814.98	-500.757	10.7374	9.86960	100	200	1954.31	39.1622	32.5918	10.3235
100	400	3609.37	-1067.70	10.7373	9.86960	100	400	3715.45	-489.800	32.5712	10.3234
200	12	126.856	46.5914	10.7623	9.86979	200	12	145.909	128.121	43.4276	10.3444
200	25	221.883	38.4751	10.7332	9.86951	200	25	296.974	222.296	36.3634	10.3280
200	50	375.093	33.5345	10.7273	9.86954	200	50	569.074	317.830	33.1587	10.3247
200	100	672.631	32.4336	10.7258	9.86957	200	100	1043.83	351.482	32.6787	10.3239
200	200	1270.65	32.4396	10.7255	9.86960	200	200	1798.21	301.340	32.5905	10.3237
200	400	2470.75	32.5771	10.7254	9.86960	200	400	3032.83	253.257	32.5700	10.3236
400	12	127.669	47.1367	10.7591	9.86998	400	12	145.951	128.207	43.4336	10.3444
400	25	214.309	39.4662	10.7302	9.86961	400	25	297.332	222.898	36.3598	10.3281
400	50	298.072	35.7842	10.7242	9.86959	400	50	571.365	320.542	33.1555	10.3247
400	100	382.843	35.1938	10.7228	9.86959	400	100	1052.75	358.556	32.6778	10.3239
400	200	528.749	35.0944	10.7224	9.86960	400	200	1775.45	312.456	32.5898	10.3237
400	400	830.244	35.0712	10.7223	9.86960	400	400	2585.90	273.140	32.5694	10.3237
800	12	128.793	47.2297	10.7582	9.87003	800	12	145.972	128.229	43.4351	10.3444
800	25	219.733	39.3946	10.7294	9.86963	800	25	297.520	223.048	36.3589	10.3281
800	50	294.869	35.7538	10.7235	9.86960	800	50	572.771	321.210	33.1547	10.3248
800	100	259.234	35.1931	10.7220	9.86961	800	100	1061.96	360.085	32.6776	10.3239
800	200	26.8611	35.0915	10.7217	9.86960	800	200	1818.64	313.304	32.5897	10.3237
800	400	-473.831	35.0680	10.7216	9.86960	800	400	2637.25	273.517	32.5692	10.3237
1600	12	129.372	47.2498	10.7580	9.87005	1600	12	145.980	128.234	43.4354	10.3444
1600	25	224.467	39.3541	10.7292	9.86964	1600	25	297.600	223.086	36.3586	10.3281
1600	50	320.136	35.7211	10.7233	9.86961	1600	50	573.388	321.376	33.1545	10.3248
1600	100	353.901	35.1755	10.7218	9.86961	1600	100	1066.49	360.451	32.6775	10.3239
1600	200	311.740	35.0756	10.7215	9.86960	1600	200	1848.64	313.380	32.5896	10.3237
1600	400	273.757	35.0524	10.7214	9.86960	1600	400	2800.77	273.375	32.5692	10.3237
exact		267.383	35.0385	10.7213	9.86960	exact		2559.61	264.323	32.5625	10.3237

in accordance with the observations made in that study; however different solution strategies were needed for this investigation, due to the more complex nature of the problem. In spite of the superior convergence rates seen for the Integral Transform solution, numerical hindrances were seen. This indicates that there is a clear need for further developments. One alternative to handle the encountered obstacles would be to apply ideas traditionally used in discrete approaches to GITT solutions, or even use a hybrid discrete-spectral methodology.

Table 4. Nusselt numbers for Hagen-Poiseuille flow (FVM).

$\mathrm{Pe}_H = 10$						$\mathrm{Pe}_H = 1$					
I	J	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$	I	J	$\xi = 0.001$	$\xi = 0.01$	$\xi = 0.1$	$\xi = 1$
12	12	142.762	107.173	-20.8444	7.73673	12	12	144.438	123.545	30.9689	8.45042
12	25	297.397	219.275	-57.1762	7.75869	12	25	298.694	233.484	4.57372	8.43917
12	50	594.589	434.854	-126.822	7.76699	12	50	594.554	439.571	-48.4931	8.43693
12	100	1188.88	866.008	-266.010	7.77077	12	100	1186.24	852.247	-151.747	8.43643
12	200	2377.44	1728.32	-544.336	7.77431	12	200	2369.72	1679.08	-355.683	8.43637
12	400	4754.48	3452.93	-1100.96	7.77433	12	400	4736.78	3333.82	-762.009	8.43634
25	12	138.154	70.5440	8.25530	7.77968	25	12	144.190	122.079	38.4667	8.46242
25	25	286.594	133.459	8.25791	7.79041	25	25	295.355	208.840	32.7172	8.45054
25	50	571.951	255.049	8.25796	7.79382	25	50	580.491	333.474	29.8808	8.44809
25	100	1142.62	498.577	8.25787	7.79521	25	100	1148.14	563.546	29.2981	8.44750
25	200	2283.95	985.823	8.25783	7.79638	25	200	2283.56	1025.35	29.1925	8.44737
25	400	4566.56	1960.41	8.25782	7.79640	25	400	4555.12	1954.67	29.1697	8.44734
50	12	131.320	33.5192	8.16864	7.77186	50	12	144.287	123.483	38.7984	8.46474
50	25	268.013	32.1138	8.17273	7.77924	50	25	294.343	206.831	32.5424	8.45274
50	50	530.943	30.9288	8.17298	7.78115	50	50	568.416	275.704	29.6815	8.45025
50	100	1056.99	30.2176	8.17295	7.78184	50	100	1096.25	296.966	29.2280	8.44964
50	200	2109.23	29.8355	8.17293	7.78225	50	200	2142.07	287.711	29.1471	8.44949
50	400	4213.78	29.6383	8.17293	7.78226	50	400	4234.42	275.353	29.1283	8.44946
100	12	124.318	26.3604	8.15113	7.75112	100	12	144.413	124.753	38.9017	8.46529
100	25	240.174	-7.20932	8.15516	7.75768	100	25	294.764	213.912	32.5456	8.45326
100	50	460.594	-74.3350	8.15541	7.75912	100	50	564.791	291.872	29.6871	8.45076
100	100	902.173	-205.750	8.15538	7.75952	100	100	1051.16	266.854	29.2510	8.45015
100	200	1786.32	-465.961	8.15536	7.75971	100	200	1947.31	37.1027	29.1711	8.44999
100	400	3555.26	-984.789	8.15535	7.75973	100	400	3702.74	-479.736	29.1525	8.44996
200	12	121.577	37.0201	8.14622	7.73862	200	12	144.495	125.246	38.9261	8.46542
200	25	213.013	30.6256	8.15028	7.74506	200	25	295.336	217.470	32.5376	8.45338
200	50	362.802	26.9930	8.15054	7.74637	200	50	566.692	310.593	29.6798	8.45088
200	100	655.213	26.2346	8.15051	7.74670	200	100	1039.82	343.126	29.2509	8.45027
200	200	1242.91	26.2734	8.15049	7.74680	200	200	1791.34	294.025	29.1718	8.45012
200	400	2421.90	26.3936	8.15049	7.74682	200	400	3021.41	247.072	29.1533	8.45008
400	12	122.538	37.7968	8.14498	7.73378	400	12	144.540	125.338	38.9321	8.46545
400	25	204.593	31.9331	8.14906	7.74022	400	25	295.707	218.081	32.5350	8.45341
400	50	283.688	29.2716	8.14932	7.74150	400	50	569.023	313.298	29.6775	8.45091
400	100	365.713	28.8534	8.14929	7.74181	400	100	1048.82	350.119	29.2505	8.45030
400	200	508.964	28.7825	8.14927	7.74189	400	200	1768.74	304.978	29.1716	8.45015
400	400	804.824	28.7659	8.14926	7.74191	400	400	2575.70	266.582	29.1532	8.45011
800	12	123.951	37.9440	8.14467	7.73227	800	12	144.562	125.361	38.9336	8.46546
800	25	210.603	31.9672	8.14875	7.73872	800	25	295.902	218.234	32.5343	8.45342
800	50	280.516	29.3177	8.14901	7.73999	800	50	570.452	313.963	29.6769	8.45092
800	100	242.552	28.9124	8.14898	7.74029	800	100	1058.09	351.632	29.2504	8.45030
800	200	14.2330	28.8384	8.14897	7.74037	800	200	1812.00	305.826	29.1716	8.45015
800	400	-475.662	28.8212	8.14896	7.74039	800	400	2627.15	266.967	29.1532	8.45011
1600	12	124.687	37.9775	8.14459	7.73185	1600	12	144.572	125.366	38.9340	8.46546
1600	25	215.905	31.9558	8.14868	7.73829	1600	25	295.984	218.272	32.5342	8.45342
1600	50	307.048	29.3076	8.14894	7.73957	1600	50	571.079	314.129	29.6767	8.45092
1600	100	338.681	28.9122	8.14891	7.73987	1600	100	1062.65	351.995	29.2504	8.45031
1600	200	298.107	28.8392	8.14889	7.73995	1600	200	1842.04	305.906	29.1716	8.45015
1600	400	261.858	28.8221	8.14888	7.73996	1600	400	2790.44	266.832	29.1532	8.45012

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