

## STABILIZED FINITE ELEMENT FORMULATION FOR UPPER-CONVECTED MAXWELL FLUID FLOWS

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**Abstract.** A new finite element formulation for the viscoelastic upper-convected Maxwell equation is presented in this work. The chosen mechanical model is obtained using a multi-field formulation involving the conservation equations of mass and momentum, coupled with the upper-convected Maxwell constitutive equation. A Galerkin-least-squares-type (GLS) formulation for extra-stress, pressure and velocity ( $\boldsymbol{\tau}$ - $p$ - $\mathbf{u}$ ) as primal variables is used to approximate this model. The stabilized formulation circumvents the compatibility conditions that arise in multi-field formulation involving the finite element subspaces for stress-velocity and pressure-velocity – the latter, known as the Babuška-Brezzi condition. Hence, any combination of finite elements is allowed in the numerical approximations herein undertaken, simplifying in this way the computational implementation of the stabilized method. The formulation is tested by analyzing the flow of an upper-convected Maxwell fluid around a cylinder between two parallel plates. In all computations, an equal-order bi-linear Lagrangian interpolations ( $Q1/Q1/Q1$ ) is used to approximate extra-stress, pressure and velocity fields. A range of Deborah numbers from zero to one is analyzed. The numerical results show good agreement with the expected features of a GLS-like formulation, generating stable and physically comprehensive approximations for all the three primal fields.

**Keywords:** Viscoelastic fluids, upper-convected Maxwell model, stabilized multi-field formulation, Galerkin least-squares method.

### 1. INTRODUCTION

A large number of fluids found in engineering applications, such as polymer melts, paints, food and cosmetic products, and drilling fluids, present a non-Newtonian fluid behavior. They may exhibit features such as shear-thinning or shear-thickening, viscoplasticity, normal stress differences in shearing flows, extension hardening and elasticity response - the so-called memory effects (Phan-Thien (2002)).

Computational fluid dynamics has ever been a powerful tool for solving non-Newtonian flow problems (Owens and Phillips (2002), for example), even though it still faces considerable difficulties. From the mechanical standpoint, the gap between the real behavior of non-Newtonian materials and the constitutive theory for their representation, may exclude the generalization of many rheological models and compromise the realism of the fluid dynamics simulations (see, for instance, Barnes (1999) and references therein).

Multi-field models consist of variational formulations for the momentum and mass governing equations coupled with an extra-stress-rate-type constitutive equation. Regarding the numerical approximations for these multi-field problems, an additional difficulty arises: the handling of the extra-stress tensor as a primal variable. In the finite element context, two compatibility conditions appear for such models: the need to satisfy the classical Babuška-Brezzi condition involving the finite element sub-spaces for velocity and pressure fields and a second compatibility condition between the extra stress and velocity finite sub-spaces.

The aim of the present article is the investigation of the numerical features of a stabilized multi-field formulation for extra stress, pressure and velocity (referred hereafter simply as  $\boldsymbol{\tau}$ - $p$ - $\mathbf{u}$ ), for the approximation of non-linear viscoelastic fluid flows. This formulation is a Galerkin-least-squares (GLS)-type method, developed as an attempt to enhance the stability of the classical Galerkin approximation for viscoelastic flows. Moreover, it circumvents the compatibility conditions between the finite sub-spaces for velocity-pressure and extra-stress-velocity fields. In addition, due to an appropriate design of its least-squares mesh-dependent terms, this formulation has the capability to remain stable even in locally advective-dominated flows, for which the inertia terms of the momentum equations play a relevant role.

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Some two-dimensional steady flow simulations of an upper-convected Maxwell fluid are performed. The flow domain is a planar channel with a confined cylinder. The ratio between the channel height and the cylinder diameter is fixed as two. The inertia is neglected and elastic effects are evaluated for a Deborah number range from zero and one. All the numerical results proved to be physically meaningful and in accordance with the related literature.

## 2. MECHANICAL MODELING

A multi-filed mechanical model, for which the primal unknown variables were the velocity,  $\mathbf{u}$ , the hydrodynamic pressure,  $p$ , and the elastic extra-stress tensor,  $\boldsymbol{\tau}$ , has been considered in this article. The fluid domain  $\Omega$  was supposed to be an open bounded subset of  $\mathfrak{R}^2$ , with a regular polygonal boundary  $\Gamma$ , such that

$$\begin{aligned} \Gamma &= \overline{\Gamma_g^u \cup \Gamma_g^\tau \cup \Gamma_h} \\ \Gamma_g^u \cap \Gamma_g^\tau \cap \Gamma_h &= \emptyset \end{aligned} \quad (1)$$

with  $\Gamma_g^u \neq \emptyset$  and  $\Gamma_g^\tau \neq \emptyset$  and the subscripts  $g$  and  $h$  standing for the portions of  $\Gamma$  on which Dirichlet and Neumann boundary conditions have been respectively applied.

Hence, from the domain definitions introduced by Eq. (1), the governing equations for the non-linear viscoelastic fluid flows herein investigated may be built with the continuity equation for incompressible materials and momentum balance equation for a continuous body undergoing a steady motion (Gurtin, 1981),

$$\begin{aligned} \rho[\nabla \cdot \mathbf{u}] - \text{div } \mathbf{T} &= \mathbf{b} \\ \text{div } \mathbf{u} &= 0 \end{aligned} \quad (2)$$

where  $\rho$  is the fluid density,  $\mathbf{T}$  is the Cauchy stress tensor and  $\mathbf{b}$  is the vector of external forces per unit of mass.

The third equation to be added to governing equations defined by Eq. (2) is a constitutive law for representation of internal stresses in the fluid. In this article, it has been assumed that the tensor  $\mathbf{T}$  may be decomposed in a spherical and a deviator portions, *i.e.*,  $\mathbf{T} = -p\mathbf{1} + \boldsymbol{\tau}$  (Gurtin, 1981). The stress deviator tensor  $\boldsymbol{\tau}$  is described by the Maxwell-B model (Astarita and Marrucci, 1974), a rate-type non-linear viscoelastic constitutive equation, which is given by:

$$\begin{aligned} \mathbf{T} &= -p\mathbf{1} + \boldsymbol{\tau} \\ \boldsymbol{\tau} + \lambda \overset{\sim}{\boldsymbol{\tau}} &= +2\eta_p \mathbf{D} \end{aligned} \quad (3)$$

where  $\mathbf{1}$  is the unity tensor,  $\eta_p$  is the polymeric viscosity,  $\mathbf{D} = 1/2(\nabla(\mathbf{u}) + \nabla(\mathbf{u}^T))$  is the rate-of-strain tensor and  $\lambda$  is the relaxation time of the material. Besides, the symbol  $\overset{\sim}{\boldsymbol{\tau}}$  stands for the upper-convected time derivative of the tensor  $\boldsymbol{\tau}$ ,

$$\overset{\sim}{\boldsymbol{\tau}} = \frac{D\boldsymbol{\tau}}{Dt} + [\nabla \boldsymbol{\tau}] \mathbf{u} - [\nabla \mathbf{u}] \boldsymbol{\tau} - [\boldsymbol{\tau}] \nabla \mathbf{u}^T \quad (4)$$

The upper-convected Maxwell model defined by Eq. (3)-(4) is a particularization of the Oldroyd-B model, if the solvent viscosity is set to zero (Astarita and Marrucci, 1974) – some times also referred simply by Maxwell-B. Despite some features of the Maxwell-B model, which prevent it from modeling the behavior of real fluids, it is widely used in computational fluid mechanics applications where the elastic effects are to be studied independently of the viscosity changes effects (Owens and Phillips (2002).

The boundary conditions that compose the mechanical model defined by Eq. (1) may be of three different types: prescribed velocity at in- and out-flow boundaries, prescribed traction and prescribed elastic stress at inflow boundaries in order to satisfy the need of the model for information about the history of stress. Thus, combining the balance and material equations defined by Eq. (2) and Eq. (3)-(4), respectively, with the appropriate velocity and stress boundary conditions, a multi-filed boundary-value problem for steady-state flows of upper-convected Maxwell viscoelastic materials may be stated as:

$$\begin{aligned}
 \rho[\nabla \mathbf{u}]\mathbf{u} &= -\nabla p + \text{div } \boldsymbol{\tau} + \mathbf{b} && \text{in } \Omega \\
 \boldsymbol{\tau} + \lambda([\nabla \boldsymbol{\tau}]\mathbf{u} - [\nabla \mathbf{u}]\boldsymbol{\tau} - [\boldsymbol{\tau}]\nabla \mathbf{u}^T) &= 2\eta_p \mathbf{D}(\mathbf{u}) && \text{in } \Omega \\
 \text{div } \mathbf{u} &= 0 && \text{in } \Omega \\
 \mathbf{u} &= \mathbf{u}_g && \text{on } \Gamma_g^{\mathbf{u}} \\
 \boldsymbol{\tau} &= \boldsymbol{\tau}_g && \text{on } \Gamma_g^{\boldsymbol{\tau}} \\
 [\boldsymbol{\tau} - p \mathbf{I}]\mathbf{n} &= \mathbf{t}_h && \text{on } \Gamma_h
 \end{aligned} \tag{5}$$

where the variables  $\rho$ ,  $p$ ,  $\mathbf{u}$ ,  $\boldsymbol{\tau}$ ,  $\lambda$ ,  $\eta_p$ ,  $\mathbf{D}$  and  $\mathbf{b}$  are defined as before,  $\mathbf{t}_h$  is the stress vector and  $\mathbf{u}_g$  and  $\boldsymbol{\tau}_g$  are the imposed velocity and extra-stress boundary conditions, respectively.

### 3. FINITE ELEMENTS APPROXIMATION

In this section, it is introduced a stabilized multi-field finite element formulation for inertia flows of upper-convected Maxwell fluids. Such a formulation employs, besides the usual finite element approximations for the pair velocity,  $\mathbf{u}$ , and pressure,  $p$ , the extra-stress tensor  $\boldsymbol{\tau}$  as primal variables.

#### 3.1. Some notation

A partition  $\Omega^h$  of  $\bar{\Omega}$  into finite elements is performed in the usual way: no overlapping is allowed between any two elements and the union of all element domains reproduces  $\bar{\Omega}$  and a combination of triangles and quadrilaterals, for the two-dimensional case, may be accommodated. Quasi-uniformity is not assumed (Ciarlet, 1978).

As usual,  $C^0(\Omega)$  stand for the space of continuous functions on  $\Omega$ ,  $L^2(\Omega)$  and  $L_0^2(\Omega)$ , and,  $H^1(\Omega)$  and  $H_0^1(\Omega)$ , Hilbert and Sobolev functional spaces, respectively, as follows (Rektorys (1975),

$$\begin{aligned}
 L_2(\Omega) &= \{q \mid \int_{\Omega} q^2 d\Omega < \infty\} \\
 L_2^0(\Omega) &= \{q \in L_2(\Omega) \mid \int_{\Omega} q d\Omega = 0\} \\
 H^1(\Omega) &= \{v \in L_2(\Omega) \mid \partial_{x_i} v \in L^2(\Omega), i=1, N\} \\
 H_0^1(\Omega) &= \{v \in H^1(\Omega) \mid v=0 \text{ on } \Gamma_g, i=1, N\}
 \end{aligned} \tag{6}$$

The operators  $(\cdot, \cdot)$  and  $\|\cdot\|$  represent the  $L^2$ -inner product and  $L^2$ -norm on  $\Omega$ , and  $(\cdot, \cdot)_K$  the  $L^2$ -inner product on  $K$ -element domain. Furthermore, one assumes that  $R_k$ , denotes the polynomial of degree  $k$  and  $R_k(K)=P_k(K)$ , if  $K$ -element is a triangle, or  $R_k(K)=Q_k(K)$ , if the  $K$ -element is a quadrilateral (Ciarlet, 1978).

#### 3.2 A multi-field stabilized formulation

Introducing the definitions of finite element sub-spaces for extra-stress, pressure and velocity as follows,

$$\begin{aligned}
 \boldsymbol{\Sigma}^h &= \{\mathbf{S} \in C^0(\Omega)^{N \times N} \cup L_2(\Omega)^{N \times N} \mid S_{ij} = S_{ji}, i, j=1, N \mid S_K \in R_k(K)^{N \times N}, K \in \Omega^h\} \\
 P^h &= \{q \in C^0(\Omega) \cup L_2^0(\Omega) \mid q_K \in R_l(K), K \in \Omega^h\} \\
 \mathbf{V}^h &= \{\mathbf{v} \in H_0^1(\Omega)^N \mid \mathbf{v}_K \in R_m(K)^N, K \in \Omega^h\} \\
 \mathbf{V}_g^h &= \{\mathbf{v} \in H^1(\Omega)^N \mid \mathbf{v}_K \in R_m(K)^N, K \in \Omega^h, \mathbf{v} = \mathbf{u}_g \text{ on } \Gamma_g\}
 \end{aligned} \tag{7}$$

a multi-field Galerkin least-squares-type formulation, for upper-convected Maxwell fluid flows, may be stated as: find the triple  $(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h) \in \boldsymbol{\Sigma}^h \times P^h \times \mathbf{V}_g^h$  such as:

$$B(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h; \mathbf{S}^h, q^h, \mathbf{v}^h) = F(\mathbf{S}^h, q^h, \mathbf{v}^h) \quad \forall (\mathbf{S}^h, q^h, \mathbf{v}^h) \in \boldsymbol{\Sigma}^h \times P^h \times \mathbf{V}^h \tag{8}$$

where

$$\begin{aligned}
 B(\boldsymbol{\tau}^h, p^h, \mathbf{u}^h; \mathbf{S}^h, q^h, \mathbf{v}^h) = & \frac{1}{2\eta_p} \int_{\Omega} \boldsymbol{\tau}^h \cdot \mathbf{S}^h d\Omega + \frac{1}{2\eta_p} \int_{\Omega} \lambda ([\nabla \boldsymbol{\tau}^h] \mathbf{u}^h - [\nabla \mathbf{u}^h] \boldsymbol{\tau}^h - [\boldsymbol{\tau}^h] \nabla \mathbf{u}^{h^T}) \cdot \mathbf{S}^h d\Omega \\
 & - \int_{\Omega} \mathbf{D}(\mathbf{u}^h) \cdot \mathbf{S}^h d\Omega + \int_{\Omega} \rho [\nabla \mathbf{u}^h] \mathbf{u}^h \cdot \mathbf{v}^h d\Omega + \int_{\Omega} 2\eta_s \mathbf{D}(\mathbf{u}) + \int_{\Omega} \boldsymbol{\tau} \cdot \mathbf{D}(\mathbf{v}^h) d\Omega - \int_{\Omega} p \operatorname{div} \mathbf{v}^h d\Omega + \epsilon \int_{\Omega} p^h q^h d\Omega \\
 & - \int_{\Omega} \operatorname{div} \mathbf{u}^h q^h d\Omega + \sum_{K \in \Omega^h} \int_{\Omega_K} (\rho [\nabla \mathbf{u}^h] \mathbf{u}^h + \nabla p^h - 2\eta_s \operatorname{div} \mathbf{D}(\mathbf{u}) - \operatorname{div} \boldsymbol{\tau}) \cdot \\
 & \quad \cdot \alpha(\operatorname{Re}_K) (\rho [\nabla \mathbf{v}^h] \mathbf{u}^h - \nabla q^h + 2\eta_s \operatorname{div} \mathbf{D}(\mathbf{v}) + \operatorname{div} \mathbf{S}^h) d\Omega \quad (9) \\
 & + 2\eta_p \beta \int_{\Omega} \left( \frac{1}{2\eta_p} \boldsymbol{\tau}^h + \frac{1}{2\eta_p} \lambda ([\nabla \boldsymbol{\tau}^h] \mathbf{u}^h - [\nabla \mathbf{u}^h] \boldsymbol{\tau}^h - [\boldsymbol{\tau}^h] \nabla \mathbf{u}^{h^T}) + \mathbf{D}(\mathbf{u}^h) \right) \cdot \\
 & \quad \cdot \left( \frac{1}{2\eta_p} \mathbf{S}^h + \frac{1}{2\eta_p} \lambda ([\nabla \mathbf{S}^h] \mathbf{u}^h - [\nabla \mathbf{u}^h] \mathbf{S}^h - [\mathbf{S}^h] \nabla \mathbf{u}^{h^T}) - \mathbf{D}(\mathbf{v}^h) \right) d\Omega
 \end{aligned}$$

and

$$\begin{aligned}
 F(\mathbf{S}^h, q^h, \mathbf{v}^h) = & \int_{\Omega} \mathbf{f} \cdot \mathbf{v}^h d\Omega + \int_{\Gamma_h} \mathbf{t}_h \cdot \mathbf{v}^h d\Gamma \\
 & + \sum_{K \in \Omega^h} \int_{\Omega_K} \mathbf{f} \cdot (\alpha(\operatorname{Re}_K) (\rho [\nabla \mathbf{v}^h] \mathbf{u}^h - \nabla q^h + \operatorname{div} \mathbf{S})) d\Omega \quad (10)
 \end{aligned}$$

with the stability parameter  $\alpha(\operatorname{Re}_K)$  for the motion equation given as suggested by Franca and Frey (1992),

$$\begin{aligned}
 \tau(\operatorname{Re}_K) &= \frac{h_K}{2|\mathbf{u}|_p} \xi(\operatorname{Re}_K) \\
 \xi(\operatorname{Re}_K) &= \begin{cases} \operatorname{Re}_K, & 0 < \operatorname{Re}_K < 1 \\ 1, & \operatorname{Re}_K > 1 \end{cases} \\
 \operatorname{Re}_K &= \frac{\rho m_k |\mathbf{u}|_p h_K}{4\eta(\dot{\gamma})} \\
 m_k &= \min\{1/3, 2C_k\} \\
 C_k &= \sum_{K \in \Omega^h} h_K^2 \|\operatorname{div} \boldsymbol{\tau}^h\|_{0,K}^2 \geq \|\boldsymbol{\tau}^h\|_K^2 \quad \forall \boldsymbol{\tau}^h \in \boldsymbol{\Sigma}^h \quad (11)
 \end{aligned}$$

and the stability parameter  $\beta$  for the viscoelastic material equation taken as 0.25, as suggested in Behr et al. (1993).

Remarks:

1. Franca and Stenberg (1991) proposed a three-field stabilized formulation for inertialess flows of Newtonian fluids. The authors have also established convergence and stability properties for the proposed formulation.
2. Behr et al. (1993) improved the results of Franca and Stenberg (1991), introducing a similar stabilized formulation but also incorporating the inertia terms. Furthermore, the authors employed a design for stability parameter incorporating the influence of the local Reynolds number and the mesh size parameter,  $h_K$  - as it has already been done in Franca and Frey (1992), for mixed approximations for constant-viscosity fluid flows.
3. The differences between the formulation proposed by Bonvin et al. (2001) and the one defined by Eq. (8)-(11) are the design of the stability parameter, Eq. (11), the presence of inertia terms and also definitions of the finite element sub-spaces for the primal variables, Eq. (7), which in this article comprehend not only triangular elements, as it is considered in Bonvin et al. (2001).

### 3.3 Matrix problem

In this section, the matrix problem associated to stabilized formulation defined by Eq. (8)-(11) is presented. Performing the finite element interpolations for trial and test functions involved in Eq. (8)-(11), the following residual system of nonlinear of algebraic equation may be obtained,

$$\mathbf{R}(\mathbf{U}) = \mathbf{0} \quad (12)$$

where  $\mathbf{U}$  is a vector formed by the degrees-of-freedom of extra-stress, pressure and velocity, associated to all nodal points of the finite element mesh,  $\mathbf{U} = ([\boldsymbol{\tau}], [p], [\mathbf{u}])^T$ , and the residual  $\mathbf{R}(\mathbf{U})$  of Eq. (11) is given by

$$\begin{aligned}
 \mathbf{R}(\mathbf{U}) = & [(\mathbf{E}) + \mathbf{E}_\tau(\mathbf{u}, \eta(\dot{\gamma})) + \mathbf{J}] \boldsymbol{\tau} + [\mathbf{N}(\mathbf{u}) + \mathbf{N}_\alpha(\mathbf{u}, \eta(\dot{\gamma})) + \mathbf{K}_\alpha(\mathbf{u}, \eta(\dot{\gamma})) + \mathbf{J}^T - \mathbf{G}^T] \mathbf{u} + [\mathbf{G} + \mathbf{G}_\alpha(\mathbf{u}, \eta(\dot{\gamma}))] p \\
 & - [\mathbf{H} + \mathbf{H}_\alpha(\mathbf{u}^h, \eta(\dot{\gamma}))] \quad (13)
 \end{aligned}$$

where  $[\mathbf{J}]$  e  $[\mathbf{J}^T]$  are matrices derived from the viscoelastic material equation due to the linking between  $\mathbf{D}$  and  $\mathbf{u}$ ,  $[\mathbf{E}]$  derived from the upper-convected derivative of tensor  $\boldsymbol{\tau}$ ,  $[\mathbf{N}(\mathbf{u})]$  derived from the advective term of motion equation,  $[\mathbf{K}]$  from its diffusive term,  $[\mathbf{H}]$  from the body force,  $[\mathbf{G}]$  from the pressure term of motion equation and  $[\mathbf{G}^T]$  from the incompressibility term of continuity equation..

To solve the residual system of nonlinear equations defined by Eq. (12)-(13), a quasi-Newton method with a frozen gradient strategy is used. At each Newton iteration, the algorithm solves, for the incremental vector  $\mathbf{A}$ , the linear Jacobian system,

$$\mathbf{J}(\mathbf{U}_k) \mathbf{A}_{k+1} = \mathbf{R}(\mathbf{U}_k) \quad (14)$$

with the Jacobian matrix  $\mathbf{J}(\mathbf{U})$  given by

$$\mathbf{J}(\mathbf{U}_k) = \frac{\partial \mathbf{R}}{\partial \mathbf{U}}(\mathbf{U}_k) \quad (15)$$

and then updated the degree-of-freedom vector,

$$\mathbf{U}_{k+1} = \mathbf{U}_k + \mathbf{A}_{k+1} \quad (16)$$

Remarks:

1. The adopted convergence criterion to stop the algorithm is that the magnitude of the residual  $\mathbf{R}(\mathbf{U}^h)$  defined by Eq.(12) must be less than  $10^{-7}$ . Otherwise, the algorithm is re-started with the computation of the Jacobian system defined by Eq. (15)-(16).

2. Null extra-stress and velocity and pressure fields are employed as initial solution estimates for the quasi-Newton solver. Besides, a continuation procedure on the advective matrix of Eq. (13)-(14) is implemented in order to improve algorithm convergence..

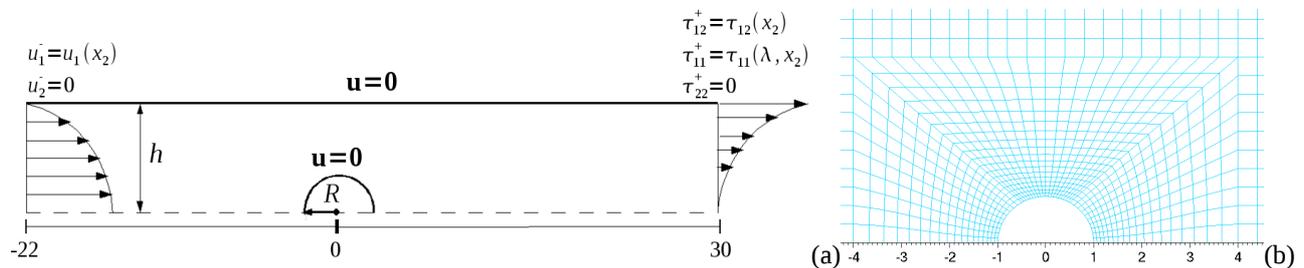


Figure 1. Flow around a cylinder kept between a channel: (a) problem statement; (b) Mesh detail around cylinder.

#### 4. NUMERICS RESULTS

The multi-field stabilized approximation for upper-convected Maxwell fluids (Eq.(8)-(11)) is tested for the flow around a cylinder between two parallel plates. Fig. 1 shows the geometry and the problem statement for a system of Cartesian coordinate with origin at the cylinder center. The channel aspect ratio is defined as the half height of the channel ( $h$ ) divided by the cylinder radius ( $R$ ) - with  $h=8m$  and  $R=1m$  - and the flow rate set as  $\bar{u}=1m/s$ . Due to symmetry and in order to reduce computational efforts, only one half of the domain has been simulated.

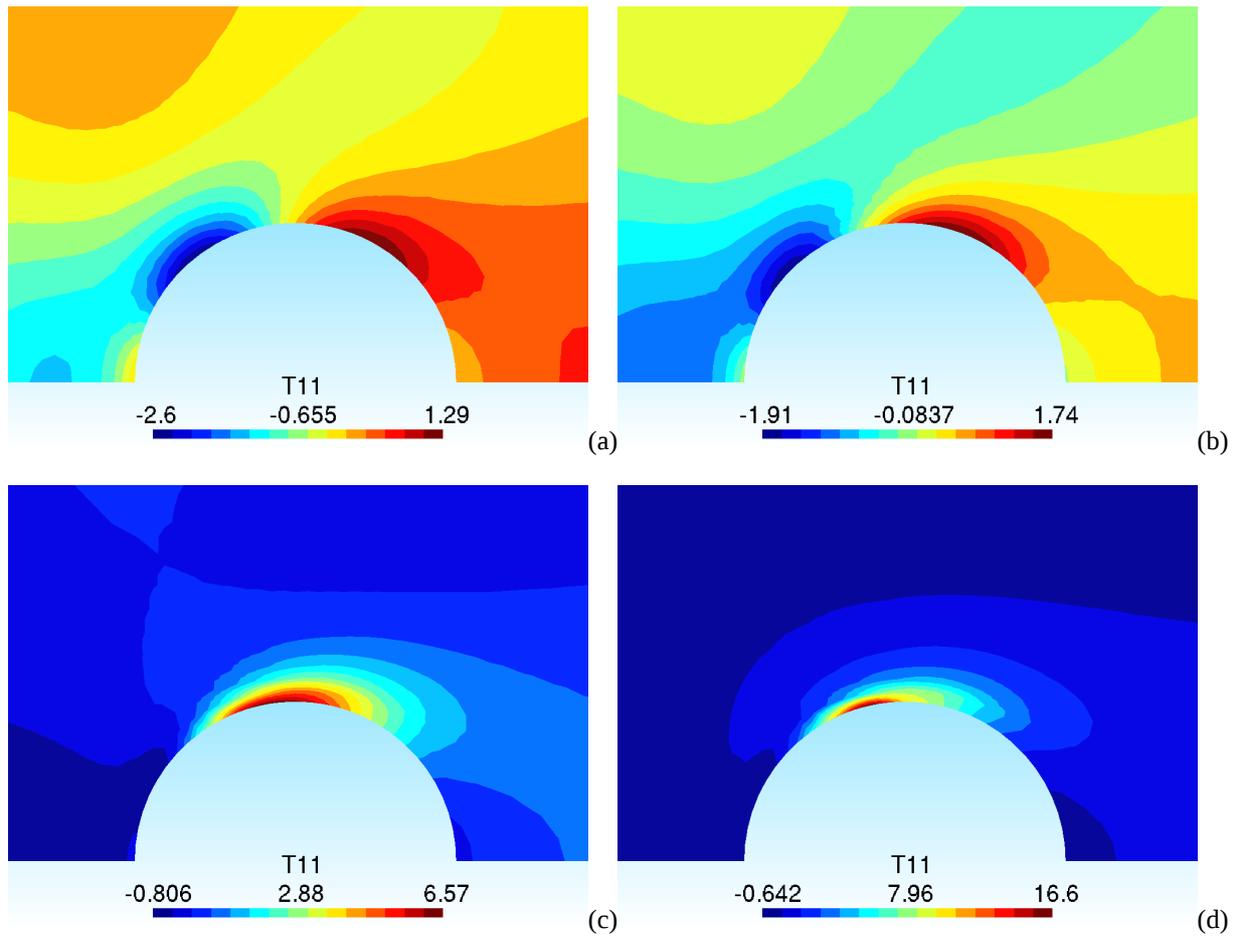
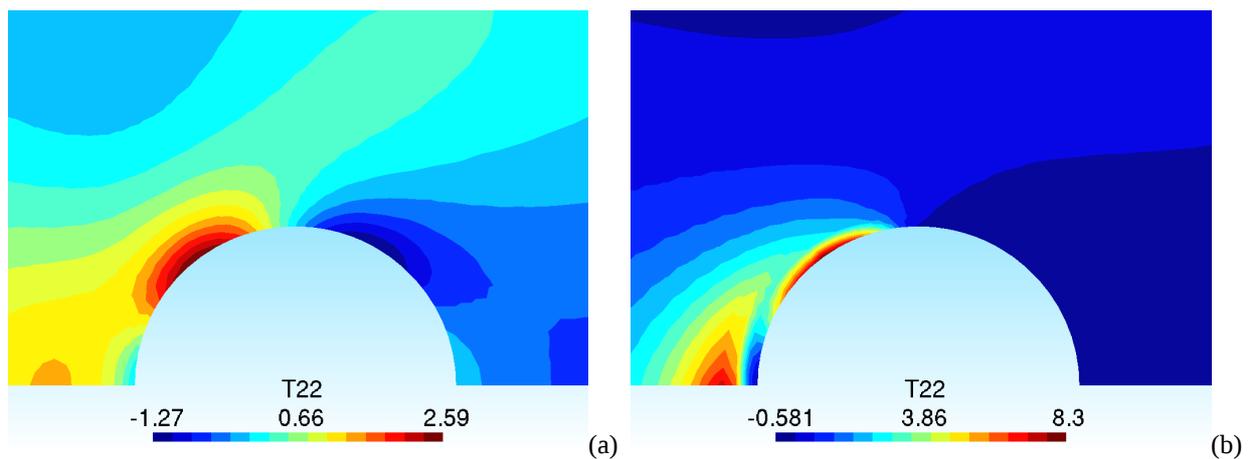


Figure 2.  $\tau_{11}$ -isobands around the cylinder, for  $Re=0$ : (a)  $De=0$ ; (b)  $De=0.1$ ; (c)  $De=0.5$ ; (d)  $De=0.8$ .

In order to partition the computational domain  $\Omega^h$  into no-overlapping quadrilateral finite elements, 25,400 quadrilateral bi-linear (Q1) elements for extra-stress, pressure and velocity – rendering a total of 131,406 degrees-of freedom – have been used – see Fig. 1a for a detail of the employed mesh in the cylinder vicinity.



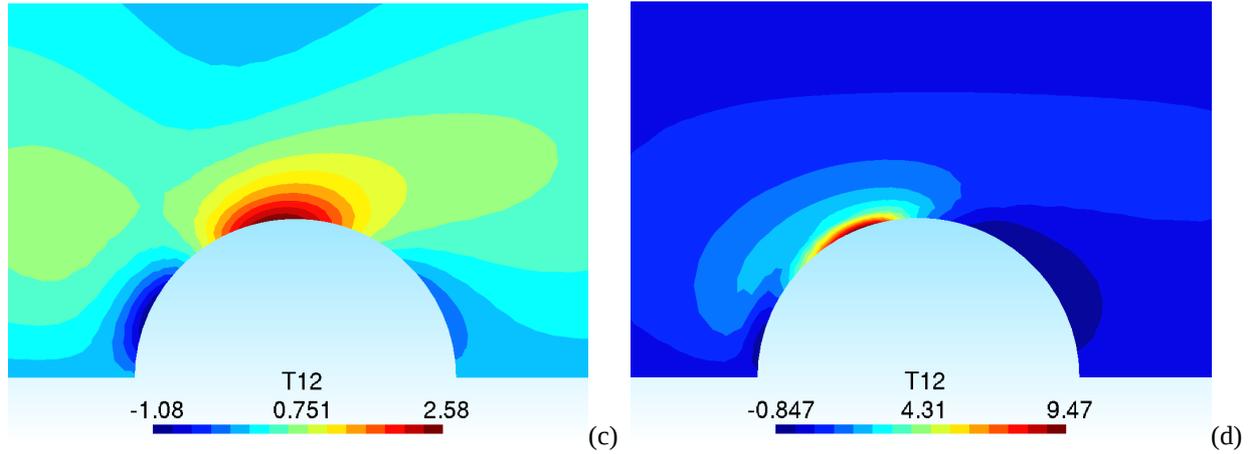


Figure 3.  $t_{22}$ - and  $t_{12}$ -isobands around the cylinder, for  $Re=0$ : (a), (c)  $De=0$ ; (b), (d)  $De=0.8$ .

The imposed boundary conditions are: no-slip and impermeability on channel walls and cylinder surface, velocity and extra-stress symmetry conditions at centerline -  $\partial x_2 u_1 = u_2 = 0 = \tau_{12} = 0$  - and fully-developed velocity extra-stress profiles at inflow and outflow (Behr *et al.*, 2004),

$$u_1^- = u_1^+ = 1.5\bar{u} (1 - x_2^2/h^2) ; u_2^- = u_2^+ = 0$$

$$\tau_{11}^- = \tau_{11}^+ = 2\lambda\mu(-3x_2/h^2)^2 ; \tau_{12}^- = \tau_{12}^+ = \eta_p(-3x_2/h^2) ; \tau_{22}^- = \tau_{22}^+ = 0$$

The Deborah number is defined as “the ratio between the fluid relaxation time and the flow characteristic time, standing for the transient nature of the flow relative to the fluid time scale” (Phan-Thien, 2002). Thus, it may be computed as

$$De = \frac{\lambda u_c}{L_c}$$

where  $\lambda$  is the relaxation time and  $u_c$  and  $L_c$  are the characteristic velocity and length, taken as the average inlet velocity,  $\bar{u}$ , and half the height of the channel,  $h$ , respectively.

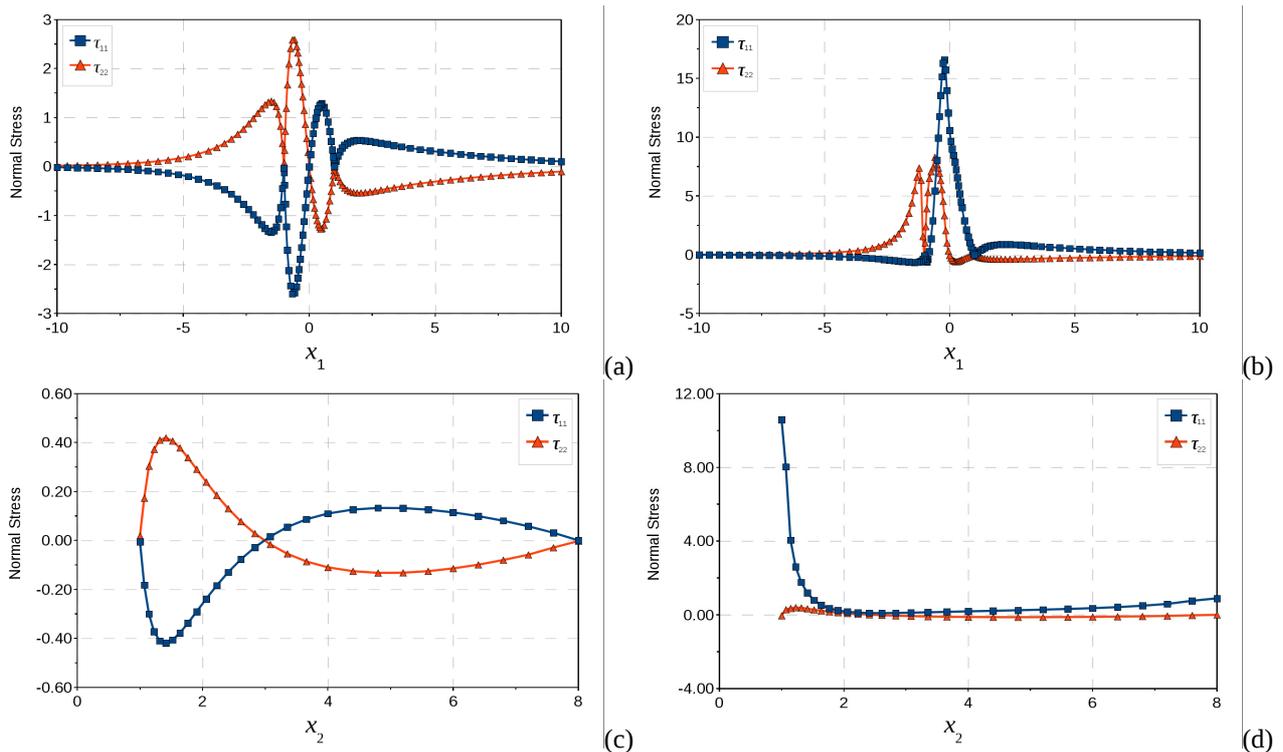


Figure 4.  $\tau_{11}$  and  $\tau_{22}$  extra-stress plotting, for  $Re=0$ : longitudinal profiles at  $x_2=0$ , for (a)  $De=0$  and (b)  $De=0.8$ ; transverse profiles at  $x_I=0$ , for (c)  $De=0$  and (d)  $De=0.8$ .

Figures 2 and 3 show normal and shear extra-stress isobands around the cylinder, for inertialess flow ( $Re=0$ ) and different Deborah numbers ( $De=0$  to  $0.8$ ). Fig. 2 illustrates  $\tau_{11}$  normal extra-stress isobands, for  $De=0$  (Fig. 2a),  $De=0.1$  (Fig. 2b),  $De=0.5$  (Fig. 2c),  $De=0.8$  (Fig. 2d), while Fig. 3 presents  $\tau_{22}$  normal and  $\tau_{12}$  shear extra-stress isobands, only for  $De=0$  (Fig. 3a-3b, for  $\tau_{22}$  isobands) and  $De=0.8$  (Fig. 3c-3d, for  $\tau_{12}$  isobands). In both figures, the influence of elastic effects introduced by UCM material model (Eq. (3)-(4)) is investigated. In Fig. 2, it can be clearly observed the dependence of  $\tau_{11}$  normal extra-stress on the Deborah number. The Newtonian case -  $De=0$  (Fig. 2a) - presents a symmetric-pattern for  $\tau_{11}$  isobands around the cylinder, a typical characteristic behavior prescribed by inelastic fluid models. This symmetry is broken as Deborah increases, with the maximum value of the extra-stress  $\tau_{11}$  reaching a value almost thirteen times greater than the Newtonian one (see Fig. 2a and 2d). Besides, this maximum normal axial traction begins to occur just before the cylinder equator for the higher values of Deborah (Fig. 2c and 2d), certainly due to the fluid extension induced by the intrusion of the cylinder into the planar channel.

Fig. 3 shows the influence of fluid elasticity on  $\tau_{22}$  and  $\tau_{12}$  around the cylinder. First, comparing the Newtonian and viscoelastic cases -  $De=0$  (Fig. 3a and Fig. 3c) and  $De=0.8$  (Fig. 3b and Fig. 3d), respectively, it can be observed that the  $\tau_{22}$  or  $\tau_{12}$  symmetrical isobands patterns have been also destroyed. It may be seen that  $\tau_{11}$  and  $\tau_{12}$  extra-stress levels increases with the elasticity, as expected of a viscoelastic fluid model. Moreover, the maximum values of the both extra-stress fields are dislocated from the cylinder equator to the inflow surface of the cylinder - with the  $\tau_{12}$  extra-stress still forming a region subjected to high values just upstream of the cylinder. Higher values of the extra-stress  $\tau_{22}$  are found before the cylinder, probably due to the need of the flow to circumvent the cylinder surface, which imposes a locally extensional kinematics to the flow leading to higher values of traction in the transverse direction to the flow.

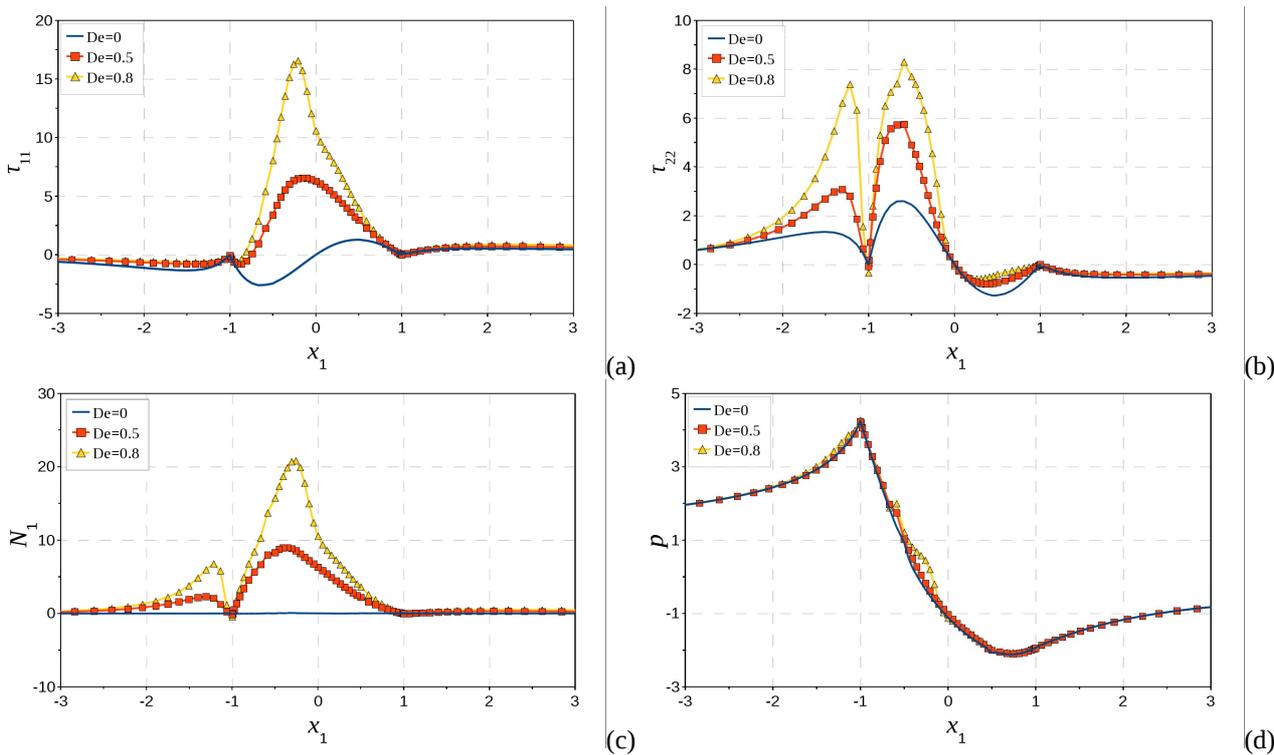


Figure 5. Longitudinal extra-stress and pressure profiles at  $x_2=0$ , for  $Re=0$  and  $De=0, 0.5$  and  $0.8$ : (a)  $\tau_{11}$ ; (b)  $\tau_{22}$ ; (c)  $N_1=\tau_{11}-\tau_{22}$ ; (d)  $p$ .

Figures 4-6 show the normal stresses and pressure profiles for inertialess flow and different Deborah numbers. It can be observed that symmetry is obtained for non-elastic case ( $De=0$ ), leading to a null first normal stress difference, as expected. As  $De$  is increased, this symmetry breaks, and the first normal stress difference departs from zero, close to the cylinder wall. Moreover, the first normal stress difference increases with the Deborah number, but the pressure distribution remains unaffected by the elasticity. Figure 6 also shows the longitudinal shear stress profile at  $x_2=h$ . It can be observed that it is independent of the Deborah number, as expected.

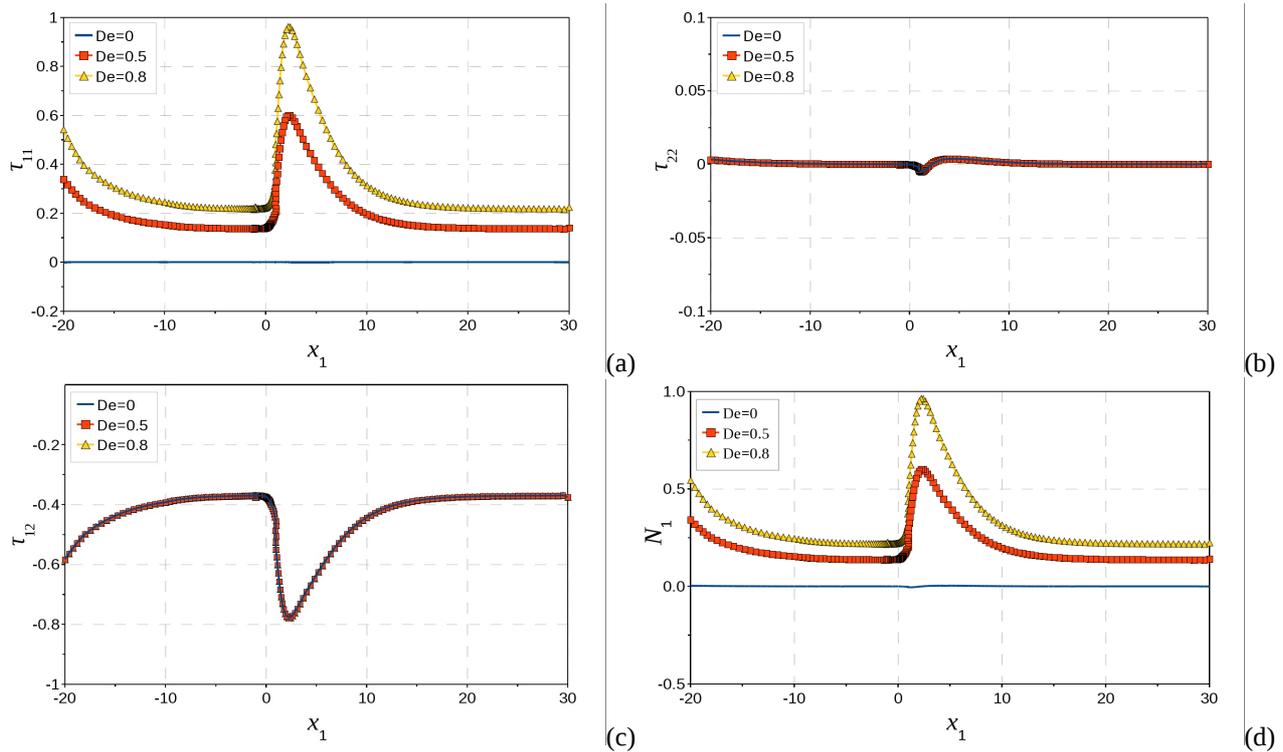


Figure 6. Longitudinal extra-stress profiles at  $x_2=h$ , for  $Re=0$  and  $De=0, 0.5$  and  $0.8$ : (a)  $\tau_{11}$ ; (b)  $\tau_{22}$ ; (c)  $\tau_{12}$ ; (d)  $N_1=\tau_{11}-\tau_{22}$ .

## 5. FINAL REMARKS

In this article, a new finite element formulation for the viscoelastic upper-convected Maxwell model is presented and tested. The stabilized formulation, a Galerkin-least-squares-type methodology, circumvents the compatibility conditions necessary in multi-field formulation. The proposed formulation is tested using the flow around a cylinder, bounded by two parallel plates. Equal-order bi-linear Lagrangian interpolations are used to approximate stresses, pressure and velocity fields. The numerical results are obtained for inertialess flows, and for a range of Deborah numbers from 0 to 0.8. The results obtained generated stable approximations for all three primal fields and are in very good agreement with the literature, indicating that the proposed formulation is promising. However, further tests should be performed.

## 6. ACKNOWLEDGEMENTS

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