TWO ALTERNATIVE APPROACHES FOR THE ANALYSIS OF STEADY STATE FLOWS IN PIPE NETWORKS

André Jacomel Torii, ajtorii@hotmail.com
Universidade Federal do Paraná
Jucélio Tomás Pereira, jucelio@utfpr.edu.br
Universidade Tecnológica Federal do Paraná
Bruno dos Santos Solheid, brunosolheid@hotmail.com
Universidade Federal do Paraná

Abstract. *The problem of steady state incompressible fluid flows in pipe networks has been studied by several authors but still represents a challenge in computational mechanics. In the present moment, two main approaches for solving this problem prevail: using the Newton method; or using the specific purpose Gradient Method. The first approach may be difficult to translate into computational routines, and may present poor convergence rates in some cases. The second approach may hide from the engineer physical insight of the problem, but is accepted as the most efficient method currently available for solving this problem. This work presents two alternative approaches for the analysis of steady state flows in pipe networks, which main goal is to simplify both computational routines and physical comprehension of the phenomenon. The first alternative approach was already studied by other authors and uses similarities between flow in pipe networks and structural analysis of bar structures. This method is based on solving the problem iteratively, using the Fixed Point Method. The second alternative approach is also based in the analogy between the flow in pipe networks and structural analysis, but this time the equations are solved incrementally. This leads to an algorithm similar to the ones used in incremental analysis for structural problems. Both methods are simple to translate into computational routines and give physical insight of the phenomenon. However, these two alternative approaches present significant differences between then. The description of these methods, its properties, advantages and draw-backs are the subject of this work. Besides, numerical examples are presented in order to validate the methods and comparisson is made to the Gradient Method.*

Keywords: *pipe network,water distribution, matrix analysis, fixed point method, incremental analysis*

1. INTRODUCTION

The analysis of steady state incompressible flows in pipe networks is a common problem in engineering practice. The solution of this problem gives the nodal pressures and internal pipes discharges of a given network, when subjected to prescribed nodal pressures and discharges. In order to build a mathematical model of this problem, it is necessary to use some formula for the pressure loss in a pipe, like the ones given by Darcy-Weisbach, for example. However, since these equations are non linear, the analysis of a network composed of several pipes leads to a system of nonlinear equations (Larock et al., 2000).

The system of nonlinear equations representing the flow in a pipe network can be solved by the Newton method, which is a general purpose method (Larock et al., 2000). However, the efficiency of the Newton method depends, in some cases, on a good initial estimative for the solution (Formiga and Chaudhry, 2008). Unfortunately, good estimative for an initial solution of the problem may be difficult to obtain, since such networks may be large and complex. Consequently, specific purpose methods for this analysis were developed. Formiga and Chaudhry (2008) present a general comparison of some commonly used methods for the analysis of pipe networks and conclude that, in most cases, the Gradient Method (Todini and Pilati, 1987) presents better convergence than other methods. Besides, the Gradient Method is implemented in most analysis packages currently available, which further encourage its use. However, the Gradient Method may hide from the engineer physical insight of the problem, since the equations used for solving the problem do not have a clear physical meaning.

The purpose of this paper is to present two alternative methods for the analysis of steady state incompressible flows in pipe networks with a clear physical meaning. The first is called here Fixed Point Method, since the problem is solved iteratively according to a general fixed point iteration. This method is also described by Kutas and Čiupailaitë (1997), but some important aspects are missing and no comparison is made with other methods. Besides, this method is also called sometimes Finite Element Approach for the analysis of pipe networks, because of its clear resemblance with the Finite Elements Methods commonly used for other problems. This name is not used here since according to the authors no reference is made to the Finite Elements Method when developing the equations. The second method described in this paper uses concepts from incremental analysis in order to solve the problem, an approach commonly used in nonlinear structural mechanics (Simo and Hughes, 1998). In this method, a relation between increments in nodal discharges and nodal pressures is written, and the problem is solved by applying small increments of nodal discharges. Results from these two methods are compared with results given by the Gradient Method (GM), which is implemented in the analysis package EPANET.

2. FIXED POINT METHOD

2.1. Matrix equations

According to Larock et al. (2000), the head loss in a pipe can be written as

$$h_f = k.q^n ,$$

where h_f is the head loss; q is the internal discharge in the pipe; and k and n are parameters which depend on the head loss formula used.

In this paper the previous equation will be rewritten as

$$h_f = k.q.\left|q\right|^{n-1}, \tag{1}$$

since a negative sign of q will represent a discharge in the opposite direction as defined in the problem, which leads to a head loss in the opposite direction. Note that a negative h_f does not mean that a head gain occurred, only that the head loss is in the opposite direction.

Table 1. Parameters for the Laminar and Darcy-Weisbach equations used in Eq. (1).

Equation	k	n	Used when
Laminar	$\frac{128.\mu.L}{\pi.\rho.D^4}$	*1*	Re < 2100
Darcy-Weisbach	$\frac{8.f.L}{\pi^2.g.D^5}$	*2*	Re ≥ 2100

The parameters k and n for the Darcy-Weisbach and Laminar formulas are presented in Tab. 1, where , μ is the fluid viscosity, ρ is the fluid density, L is the length of the pipe, D is the diameter of the pipe, g is the gravity acceleration and f is the friction factor, which according to White (2006) can be approximated by

$$\frac{1}{\sqrt{f}} \approx -1{,}8.\log\left[\frac{6{,}9}{\mathrm{Re}} + \left(\frac{k_s / D}{3{,}7}\right)^{1{,}11}\right], \tag{2}$$

where k_s is the material relative roughness and Re is the Reynolds number, which depends on the internal discharge q and is

$$\mathrm{Re} = \frac{4.\rho}{\pi.\mu.D}.q \ . \tag{3}$$

In order to solve the problem iteratively for internal discharges, Eq. (1) can be rewritten as

$$q^{(i)} = \frac{1}{k.\left|q^{(i-1)}\right|^{n-1}}.h_f^{(i)} , \tag{4}$$

where i represents the present iteration and i-1 the last one.

Equation (4) can be simplified as

$$q^{(i)} = K^{(i)}.h_f^{(i)} , \tag{5}$$

where $K^{(i)}$ is called here the *permeability coefficient* in the present iteration and is given by

$$K^{(i)} = \frac{1}{k.\left|q^{(i-1)}\right|^{n-1}} , \tag{6}$$

Note the distinction between k and K, since the first is the parameter as defined in Tab. 1 for different flow models, while the later is the *permeability coefficient* for each iteration, which takes into account both the contributions from k and $|q^{(i-1)}|^{n-1}$.

The fixed point method is an interesting approach for solving Eq. (5) iteratively, since it does not need the evaluation of the gradient of the function. Equation (5) can be solved for a given head loss by taking an initial solution for the flow $q^{(0)}$ in order to obtain the flows $q^{(1)}$, $q^{(2)}$,…, $q^{(n)}$ until convergence of the solution is achieved. Even if this approach may not be useful when dealing with a single pipe, it may simplify the analysis of systems composed of several pipes.

According to Eq. (5), the following relations can be written for nodes *1* and *2* from the pipe shown in Fig. 1.a:

$$\begin{cases} K^{(i)}.h_f^{(i)} = Q_1 \\ -K^{(i)}.h_f^{(i)} = Q_2 \end{cases} \Rightarrow \begin{cases} K^{(i)}.\left(p_2^{(i)} - p_1^{(i)}\right) = Q_1 \\ K^{(i)}.\left(p_2^{(i)} - p_1^{(i)}\right) = -Q_2 \end{cases},$$

where Q_1 is the discharge at node *1*, Q_2 is the discharge at node *2*, p_1 is the pressure at node *1*, p_2 is the pressure at node *2*. Note that from now on the uppercase Q is used for nodal discharges, while the lowercase q is used for internal pipe discharges. Besides, in the previous system of equations, the second equation is negative since Q_2 leads to a head loss in the opposite direction of that obtained when applying Q_1.

The previous relations can be written in matrix form as

$$K^{(i)}.\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.\begin{bmatrix} p_1^{(i)} \\ p_2^{(i)} \end{bmatrix} = -\begin{bmatrix} Q_1 \\ Q_2 \end{bmatrix}. \tag{7}$$

It is important to note that since several pipes can be linked to a given node, the flows Q_1 and Q_2 are not necessarily equal in magnitude. These flows actually represent the boundary conditions of the problem. Besides, Eq. (7) is similar to the equations used for the analysis of trusses or bars by matrix methods. The global matrix of the entire pipe network can be assembled by the superposition of the matrix of each pipe, as described in texts on the Finite Elements Method (Bathe, 1996). Then, it is necessary to apply the boundary conditions of the problem, by specifying the nodal discharges and at least one nodal pressure. The nodal pressures are then given by solving the resulting system of linear equations.

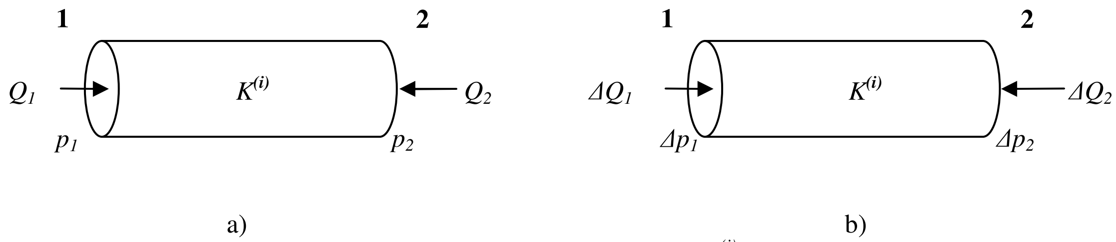


Figure 1. Pipe between nodes *1* and *2*, with a) permeability coefficient $K^{(i)}$, nodal discharges Q_1 and Q_2, and nodal pressures p_1 and p_2.b) Increment in nodal discharges ΔQ_1 and ΔQ_2, and increment in pressures Δp_1 and Δp_2.

The assemblage procedure leads to the following global equation, which holds for the entire network:

$$\mathbf{K}^{(i)}.\mathbf{p}^{(i)} = -\mathbf{Q}, \tag{8}$$

where $\mathbf{K}^{(i)}$ is the *permeability matrix* of the entire network at the current iteration, $\mathbf{p}^{(i)}$ is the vector of nodal pressures to be obtained and $\mathbf{Q}$ is the vector of applied nodal discharges. Note that the permeability matrix is analogue to the stiffness matrix; the vector of nodal pressures is analogue to the vector of nodal displacements; and the vector of nodal discharges is analogue to the vector of applied forces from structural analysis (Bathe, 1996).

The matrix $\mathbf{K}^{(i)}$ must be updated at each iteration since it depends on the pipes internal discharges $\mathbf{q}^{(i)}$, by means of the permeability coefficient from Eq. (6). The internal discharge on a pipe can be obtained once the pressures at its nodes are found, by the use of Eq. (5). Besides, since the matrix $\mathbf{K}^{(i)}$ changes continually during the analysis procedure, it is necessary to apply the pressure boundary conditions at each iteration. Note that applying pressure boundary conditions is analogue to applying displacements boundary conditions on structural problems (Bathe, 1996).

2.2. General algorithm

Considering the previous paragraphs, the following steps can be used for the analysis of a pipe network by the Fixed Point Method (FPM):

1. Start the algorithm with: pipe discharges vector $\mathbf{q}^0 = \mathbf{0}$ and a counter $i = 0$;
2. Build the vector $\mathbf{Q}$ from the nodal discharges boundary conditions;
3. Update the counter $i = i+1$;
4. Check if the flow in each pipe is laminar or turbulent and use the appropriate coefficient k and n as defined in Tab. 1 for each pipe;
5. Assemble the *permeability matrix* for the entire network $\mathbf{K}^{(i)}$, using Eq. (7) for each pipe;
6. Apply the pressure boundary conditions by assigning at least one nodal pressure;
7. Solve the system of linear equations defined by Eq. (8) thus finding the vector of nodal pressures $\mathbf{p}^{(i)}$;
8. Obtain the vector of internal flows $\mathbf{q}^{(i)}$ by applying Eq. (5) for each pipe (See next section for improvement);
9. Check the convergence of the problem. If it converges stop the procedure and take as solution the vector of nodal pressures $\mathbf{p}^{(i)}$ and the vector of pipe discharges $\mathbf{q}^{(i)}$. If the problem does not converge return to step 3.

In the previous algorithm, the convergence check of step 9 can be made on the difference between successive nodal pressures $\mathbf{p}^{(i)}$ and $\mathbf{p}^{(i-1)}$, on the pipe flows $\mathbf{q}^{(i)}$ and $\mathbf{q}^{(i-1)}$, or both vectors. Besides, on step 4 the flow can be considered laminar if the Reynolds Number is less than 2100, for example (White, 2006). Note that when the flow in a pipe is considered laminar, its permeability coefficient K as defined by Eq. (6) does not depend on the pipe discharge q, since for laminar flows n is equal to unity and thus q disappears from Eq. (6). This leads to the conclusion that pipes with discharges close or equal to zero will still have a defined value K. However, if only the expression for turbulent flows is considered, pipes with internal discharges equal to zero lead to an infinite value of K, since there will appear a division by zero in Eq. (6). Thus, in order to guarantee stability of the procedure it is necessary to use expressions for both turbulent and laminar flows.

Note also that the vector of nodal discharges $\mathbf{Q}$ will remain constant for the entire analysis procedure. However, the nodal pressures boundary conditions must be applied at each iteration since the permeability matrix $\mathbf{K}^{(i)}$ changes continually.

2.3. Improvement of the Fixed Point Method

Using the algorithm described in the last section may lead to a non monotonic convergence for the values of nodal pressures and internal discharges. The analysis of the network from Fig. 2.a, with the algorithm described in the previous section, gives the sequence of values for internal discharge and nodal pressure denoted as *original* in Fig. 3. Note that the values of internal discharges in Fig. 3.a approach the solution by alternating values above and below it. This type of convergence must be avoided, since this may lead to an algorithm which iterative solutions oscillates around the solution, but takes time to effectively approach it.

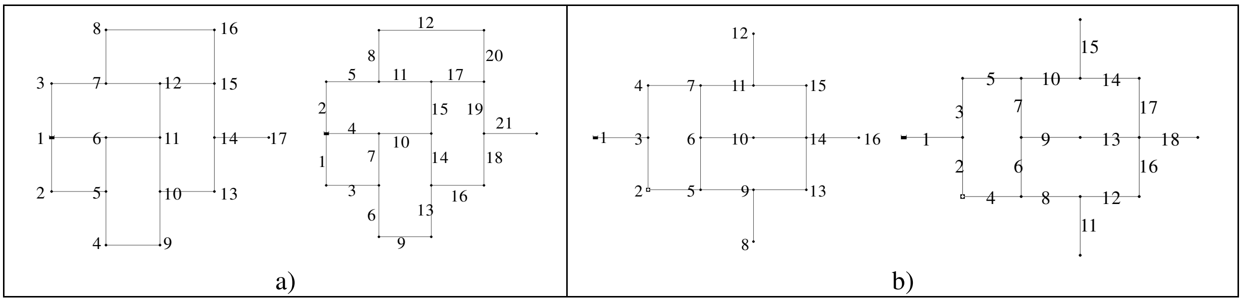
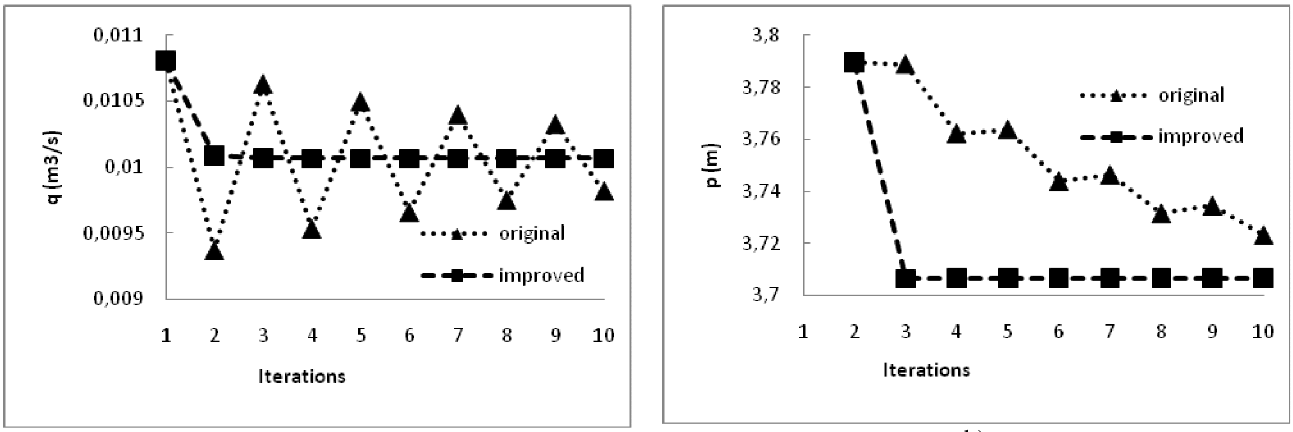


Figure 2. a) First example of a pipe network with its nodes and pipes numbered. All pipes have a length of 50m, except pipe 12 which has length of 100m. Node 1 has a prescribed pressure of 15m, while all other nodes have a demand of 5.10^{-3}m^3/s (5 L/s). b) Second example of a pipe network with its nodes and pipes numbered. All pipes have a length of 50m. Node 1 has a prescribed pressure of 15m, while nodes 8, 12 and 16 have a demand of 10.10^{-3}m^3/s (10 L/s). For both examples all pipe diameters are of 0.1m, the constant k_s is 0.046mm and the fluid is water at 15°C.

From Fig. 3, it seems that the major oscillation problem is caused by the internal discharges, since this variable presents more drastic changes. Thus, in order to avoid this type of convergence, the rule for the update of internal discharges must be reformulated. In order to achieve this, a relaxation technique is used. When updating the internal discharges, the values of the next iteration can be taken as the mean value between the values found in the current iteration, and the values from the previous iteration. Thus, step 8 from the algorithm of the FPM described in the previous section is substituted by:

8 Obtain a preliminary vector $\mathbf{q}_p^{(i)}$ by applying Eq. (5) for each pipe. If i >1 make $\mathbf{q}^{(i)}$ = ½.$\mathbf{q}_p^{(i)}$ + ½.$\mathbf{q}^{(i-1)}$. Else if i = 1 make $\mathbf{q}^{(i)} = \mathbf{q}_p^{(i)}$;

Note that the new update rule for the internal discharges is not applied for the first iteration since $\mathbf{q}^{(0)}$ is usually taken equal to a zero vector. This modification leads to an algorithm here called *Improved* Fixed Point Method (IFPM), which's sequence of values is also shown in Fig. 3. Note, that for the improved algorithm the values present monotonic convergence, as desired. Besides, it leads to a better convergence rate as shown by Fig. 3 and the examples presented in a following section.



a) b)

Figure 3. Evolution of the a) internal discharge at pipe 17 and b) pressure at node 17 for the network from Fig. 2.a; using the *original* and the *improved* Fixed Point Method.

3. INCREMENTAL METHOD

3.1. Incremental equations

Expanding Eq. (1) by a Taylor series and neglecting the second order terms gives

$$h_f(q+\Delta q) \approx h_f(q) + \Delta q.h'_f ,$$

which leads to

$$\Delta h_f \approx \Delta q.h'_f , \tag{9}$$

considering $h_f(q+\Delta q)- h_f(q) = \Delta h_f$.

Eq. (9) can be used to obtain the head loss in a pipe incrementally, since an increment in the pipe internal discharge leads to an increment in the head loss. When studying a single pipe, this approach does not present advantages in relation with the use of Eq. (1). However, in the case of a pipe network with several pipes, the use of Eq. (1) leads to a system of nonlinear equations, since n can be different from unity for turbulent flow regimes. The use of Eq. (9), instead, leads to a series of system of linear equations, since the relation between the increments of head losses and internal discharge is linear.

Before applying Eq. (9) to a given pipe it is necessary to obtain an expression which relates the increment of the head loss Δh_f in a pipe with the increment of its nodal pressures Δp_1 and Δp_2. From Fig. 1.a it can be noted that the head loss is

$$h_f = p_2 - p_1 .$$

In order to obtain Δh_f it is necessary to compare this head loss with the head loss at an previous iteration $t = 0$, for example, which is given by

$$h_f^{\,t=0} = p_2^{\,t=0} - p_1^{\,t=0} .$$

The change of the head loss in a pipe can then be written as

$$\Delta h_f = h_f - h_f^{\,t=0} = (p_2 - p_1) - (p_2^{\,t=0} - p_1^{\,t=0}) = (p_2 - p_2^{\,t=0}) - (p_1 - p_1^{\,t=0}),$$

which gives

$$\Delta h_f = \Delta p_2 - \Delta p_1 . \tag{10}$$

According to Eq. (9) and Eq. (10), the following relations can be written for an arbitrary pipe as shown in Fig. 1.b:

$$\begin{cases} \Delta h_f = \Delta Q_1 . h'_f \\ -\Delta h_f = \Delta Q_2 . h'_f \end{cases} \Rightarrow \begin{cases} \Delta p_2 - \Delta p_1 = \Delta Q_1 . h'_f \\ \Delta p_2 - \Delta p_1 = -\Delta Q_2 . h'_f \end{cases},$$

which can be written in matrix form as

$$\frac{1}{h'_f} . \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} . \begin{bmatrix} \Delta p_1 \\ \Delta p_2 \end{bmatrix} = - \begin{bmatrix} \Delta Q_1 \\ \Delta Q_2 \end{bmatrix} . \tag{11}$$

Eq. (11) holds for a pipe with initial node 1 and final node 2. Thus, in order to obtain the matrix for the entire pipe network it is necessary to assemble the matrix as discussed previously and as shown in Bathe (1996). This assemblage leads to the following equations for the entire network:

$$\mathbf{H}^{(i)} . \Delta \mathbf{p}^{(i)} = -\Delta \mathbf{Q}^{(i)} , \tag{12}$$

where the matrix **H** is assembled from each pipe, $\Delta\mathbf{p}$ is the vector of pressure increments related to a given nodal discharge increment vector $\Delta\mathbf{Q}$. Besides, note that in Eq. (12) the index *(i)* has been introduced to express the fact that the equation must be solved for each incremental step.

The solution of Eq. (12) gives the nodal pressures increments to be applied. Thus, the nodal pressures can be updated as

$$\mathbf{p}^{(i)} = \mathbf{p}^{(i-1)} + \Delta \mathbf{p}^{(i)} , \tag{13}$$

where $\mathbf{p}^{(i)}$ is the updated nodal pressure vector and $\mathbf{p}^{(i-1)}$ is the same vector in the previous iteration.

The internal pipes discharges can be updated using Eq. (9) locally by writing

$$\Delta q^e = \frac{\Delta h^e{}_f}{h^{e}{}'_f} , \tag{14}$$

where *e* represents the pipe to which Eq. (14) is applied.

Note that Eq. (14) is evaluated locally, with Δh_f being evaluated with Eq. (10), and the derivative h'_f of the head loss equations, which can be evaluated analytically. Once the increment in each pipe discharge is found, the vector of increments in pipe discharges $\Delta\mathbf{q}^{(i)}$ can be assembled. The internal discharges can then be update according to

$$\mathbf{q}^{(i)} = \mathbf{q}^{(i-1)} + \Delta \mathbf{q}^{(i)} . \tag{15}$$

In order to obtain the vector of the nodal discharges increment $\Delta\mathbf{Q}$ which is used in Eq. (12), it is necessary to divide the vector of the total applied nodal discharges by the number of incremental steps to be used *N*, which gives

$$\Delta \mathbf{Q} = \frac{\mathbf{Q}}{N} . \tag{16}$$

Note that since the error of the approximated Taylor series used in Eq. (9) decreases for lower values of Δq, the analysis will be more accurate when a higher number of incremental steps *N* is used.

3.2 Head loss formulas and its derivatives

In order to apply the incremental analysis it is necessary to define the head loss equation $h_f(q)$ and its first derivative $h'_f(q)$, since these two expressions are used in Eq. (11) and Eq.(14). As stated previously, the head loss equation is different for laminar and turbulent flow regimes. Starting by the case of a laminar flow, Tab. 1 gives the following expression for Eq. (1):

$$h_f = \frac{128.\mu.L}{\pi.\rho.D^4}.q\,, \tag{17}$$

which derivative is

$$h'_f = \frac{128.\mu.L}{\pi.\rho.D^4}\,. \tag{18}$$

For turbulent flow regimes, substituting Eq. (3) in Eq. (2) and then in Eq. (1), the Darcy-Weisbach takes the form

$$h_f = \frac{1}{\left(-1{,}8.\log\left[\frac{6{,}9.\pi.\mu.D}{4.\rho.|q|} + \left(\frac{ks/D}{3{,}7}\right)^{1{,}11}\right]\right)^2}.\frac{8.L}{\pi^2.g.D^5}.q.|q|\,, \tag{19}$$

which's first derivative can be found using a symbolic mathematical software and is

$$\begin{aligned} h'_f = & \frac{8{,}518518517.\ln(10)^2.L.q.\mu.\rho}{\ln\left(\frac{1{,}725.\pi.\mu.D}{\rho.|q|} + 0{,}2340432317.\left(\frac{k_s}{D}\right)^{1{,}11}\right)^3.\pi.g.D^4.\rho^2.|q|^2.\left(\frac{1{,}725.\pi.\mu.D}{\rho.|q|} + 0{,}2340432317.\left(\frac{k_s}{D}\right)^{1{,}11}\right)} \\ & + \frac{2{,}469135802.\ln(10)^2.L}{\ln\left(\frac{1{,}725.\pi.\mu.D}{\rho.|q|} + 0{,}2340432317.\left(\frac{k_s}{D}\right)^{1{,}11}\right)^2.\pi^2.|q|..D^4}\,. \end{aligned} \tag{20}$$

3.3 Improvement of the Incremental Method

Figure 4 shows the evolution of a nodal pressure and an internal discharge when using the *original* Incremental Method, as described in the previous section. Note that the value found for the internal discharge agrees with the value given by the GM. However, the value found for the nodal pressure is significantly different. Thus, the algorithm must be improved in order to enhance its accuracy for nodal pressure evaluations.

One approach commonly used when solving nonlinear equation by incremental methods is that of dividing each incremental step in two stages (Simo and Hughes, 1998). The first is identical to the step described in the previous section, here called IM, and is used to predict the values of the variables. Thus, it is called the *predictor* stage. The second stage, however, corrects the predicted values, and is called the *corrector* stage. The incremental procedure previously presented uses only the *predictor* stage and this can lead to error accumulation during the analysis. Consequently, it is expected that the implementation of a *corrector* stage may improve the results given by the analysis.

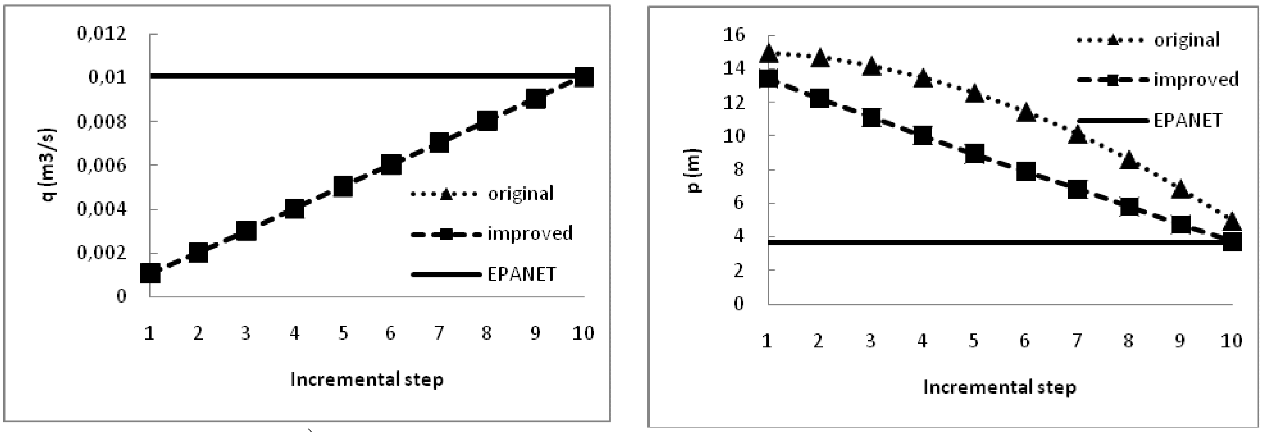
Since the *predictor-corrector* approach is used for many different types of problems, several types of *corrector* procedures can be found in literature. However, as can be seen from Fig. 4 and some following examples, the IM is not accurate for the nodal pressure calculations, while it presents good convergence rate for internal discharges. Thus, the *corrector* stage must act mainly in the evaluation of nodal pressures. Since the FPM presents good convergence rate for nodal pressures (as shown in following examples), this method can be used as the *corrector* stage for the analysis. Note that if the internal discharges are know (which is the case when the FPM is used as pressure corrector), one iteration of the FPM is sufficient to give the correct nodal pressures, since more iterations of the method are needed only when the internal discharges have to be updated.

The incremental method with pressure corrector, here called *Improved* Incremental Method (IIM) can then be summarized by the following algorithm:

1. Start the algorithm with: pipes discharges $\mathbf{q}^0=\mathbf{0}$, nodal pressures $\mathbf{p}^0=\mathbf{0}$ and a counter $i = 0$;

2. Define the vector of nodal discharges **Q**, the number of incremental steps N to be used, find the vector of nodal discharges increment $\Delta\mathbf{Q}$ with Eq. (16);
3. Update the counter $i = i+1$;
4. Start the *predictor* stage:
    a. Check if the flow in each pipe is laminar or turbulent and use Eq. (17) and Eq. (18) for pipes in laminar flow regime and Eq. (19) and Eq. (20) for pipes in turbulent flow regime;
    b. Assemble the matrix of the inverse derivatives $\mathbf{H}^{(i)}$ using Eq. (11) for each pipe;
    c. Apply the pressure boundary conditions by assigning one nodal pressure equal to zero;
    d. Solve the system of linear equations defined by Eq. (12) thus finding the vector of nodal pressure increments $\Delta\mathbf{p}^{(i)}$;
    e. Update the vector of nodal pressures $\mathbf{p}^{(i)}$ using Eq. (13);
    f. Evaluate the vector of increment in internal flows $\Delta\mathbf{q}^{(i)}$ applying Eq. (14) locally for each pipe;
    g. Update the vector of internal flows $\mathbf{q}^{(i)}$ using Eq. (15);
5. Start the *corrector* stage:
    a. Assemble the permeability matrix for the entire network $\mathbf{K}^{(i)}$, using Eq. (7) for each pipe;
    b. Apply the pressure boundary conditions by assigning one nodal pressure equal to zero;
    c. Solve the system of linear equations defined by Eq. (8) using $\mathbf{q}^{(i)}$ as the vector of independent variables, thus correcting the vector of nodal pressures $\mathbf{p}^{(i)}$;
6. Repeat steps 3 to 5 until reaching the total number of incremental steps *N*.

Results of the IIM are shown in Fig. 4 and in some folowing examples. From Fig. 4, it can be seen that the internal discharges from both *original* and *improved* methods are the same, since the *corrector* stage acts only in the nodal pressures. However, note that the nodal pressure evolution during the incremental steps are different, being the values given by the *improved* algorithm much closer to the one found using the GM.



a) b)

Figure 4. Evolution of the a) internal discharge at pipe 17 and b) pressure at node 17 from the network from Fig. 3; using the *original* and the *improved* Incremental Method. The values given by EPANET are shown asthe constant line. In a) the values of the *original* and *improved* algorithms are the same.

4. NUMERICAL EXAMPLES

The first example is the analysis of the pipe network from Fig. 2.a. This network was solved by the two alternative methods proposed, both for its *original* and *improved* versions as previously described. Besides, the example was also solved using the GM, in order to allow comparison of the methods. The results of each analysis are show in Tab. 2. Note that the dimensions used are meters (m) for head pressures and liters per second (L/s) for discharges, in order to allow an easy reading of the results. The analysis using the GM was made in the software EPANET, with an accuracy of 10^{-6} for both pressures and internal discharges. Note that this accuracy was achieved in 6 iterations, and the GM is known in literature for its high convergence rate (Formiga and Chaudhry, 2008). The analysis with the FPM, the IFPM and the IM were made using 10 iterations or incremental steps, while the analysis with the IIM was made using 5 incremental steps, since this method solves two systems of linear equations per incremental step (one for the *predictor* stage and another for the *corrector* stage).

Errors on pressures were evaluated by taking the modulus of the maximum difference between the pressures obtained with the GM and the pressures obtained with the method for which the error is evaluated. This is the same as taking the infinity norm of the difference between the vector of pressures given by the GM and the vector of pressures given by the method being compared. Consequently, the error *e* is, here, given by

$$e = \left\| \mathbf{p}_{GM} - \mathbf{p} \right\|_{\infty} = \max \left| \mathbf{p}_{GM} - \mathbf{p} \right|, \qquad (21)$$

where $\mathbf{p}_{GM}$ is the vector of pressure given by the GM and $\mathbf{p}$ is the vector of pressures given by the method for which the error is evaluated. Errors on internal discharges are evaluated in the same way.

From the results shown in Tab. 2 and Tab. 4 it can be seen that the FPM presented good results for the evaluation of nodal pressures, but poor results for the evaluation of internal discharges. The IM, instead, presented poor results for the evaluation of nodal pressures, but good results for the evaluation of internal discharges. Consequently, the IIM, which combines characteristics of both methods, presents good results for both pressures and internal discharges, as expected. The IFPM presents a better convergence rate than the FPM, for the reasons explained previously.

Table 2. Pressures and internal discharges for the example of Fig. 2.a.[1]

	Pressures				
node	**EPANET**	**FPM**	**IFPM**	**IM**	**IIM**
1	15.0000	15.0000	15.0000	15.0000	15.0000
2	11.0280	11.0364	11.0295	11.4655	11.0308
3	9.3294	9.3372	9.3320	9.9547	9.3339
4	6.3949	6.246	6.3989	7.3451	6.4019
5	8.5449	8.5609	8.5477	9.2567	8.5499
6	8.6733	8.6854	8.6758	9.3712	8.6779
7	5.4670	5.4823	5.4720	6.5200	5.4752
8	4.6337	4.6526	4.6404	5.7792	4.6440
9	5.3004	5.3231	5.3054	6.3721	5.3087
10	4.9148	4.9408	4.9203	6.0291	4.9237
11	5.4620	5.4797	5.4666	6.5157	5.4698
12	4.9863	5.0045	4.9917	6.0921	4.9951
13	4.1314	4.1559	4.1377	5.3319	4.124
14	3.9234	3.9467	3.9300	5.1459	3.9338
15	4.1637	4.1839	4.1697	5.3601	4.1733
16	4.1639	4.1865	4.1723	5.3604	4.1760
17	3.6996	3.7233	3.7066	4.9461	3.7104

	Internal discharges				
pipe	**EPANET**	**FPM**	**IFPM**	**IM**	**IIM**
1	23.0171	24.302	23.0190	23.0202	23.0102
2	27.6850	27.3255	27.6839	27.6838	27.6850
3	18.0171	19.302	18.0190	18.0202	18.0102
4	29.2979	28.2543	29.2971	29.2960	29.3049
5	22.6850	22.3255	22.6839	22.6838	22.6850
6	-16.7088	-17.7040	-16.7105	-16.7070	-16.6974
7	-3.6916	-3.2838	-3.6914	-3.6868	-3.6873
8	10.1322	10.1293	10.1280	10.1281	10.1210
9	11.7088	12.7040	11.7105	11.7070	11.6974
10	20.6063	19.9705	20.6057	20.6093	20.6176
11	7.5528	7.1962	7.5558	7.5557	7.5640
12	5.1322	5.1293	5.1280	5.1281	5.1210
13	6.7088	7.7040	6.7105	6.7070	6.6974
14	-8.0958	-7.3436	-8.0955	-8.0992	-8.1110
15	7.5105	7.6269	7.5102	7.5100	7.5066
16	9.8046	10.0476	9.8060	9.8062	9.8085
17	10.0632	9.8230	10.0660	10.0657	10.0705
18	4.8046	5.0476	4.8060	4.8062	4.8085
19	-5.1954	-4.9524	-5.1940	-5.1938	-5.1915
20	-0.1322	-0.1293	-0.1280	-0.1281	-0.1210
21	5.0000	5.0000	5.0000	5.0000	5.0000

Table 3. Pressures and internal discharges for the example of Fig. 2.b.

	Pressures				
node	**EPANET**	**FPM**	**IFPM**	**IM**	**IIM**
1	15.0000	15.0000	15.0000	15.0000	15.0000
2	6.6261	6.6297	6.6297	7.5479	6.6326
3	8.3767	8.3788	8.3788	9.1043	8.3810
4	6.6261	6.6297	6.6297	7.5479	6.6326
5	4.8755	4.8807	4.8807	5.9915	4.883
6	4.7459	4.7513	4.7514	5.8755	4.7548
7	4.8755	4.8807	4.8807	5.9915	4.883
8	3.020	3.0486	3.0486	4.3595	3.0526
9	3.8540	3.8605	3.8605	5.0828	3.8643
10	4.2805	4.2871	4.2866	5.4615	4.2903
11	3.8540	3.8605	3.8605	5.0828	3.8643
12	3.020	3.0486	3.0486	4.3595	3.0526
13	3.8346	3.827	3.822	5.0652	3.8450
14	3.8152	3.8230	3.8219	5.0475	3.8257
15	3.8346	3.827	3.822	5.0652	3.8450
16	3.0022	3.0110	3.0099	4.3243	3.0140

	Internal discharges				
pipes	**EPANET**	**FPM**	**IFPM**	**IM**	**IIM**
1	30.0000	30.0000	30.0000	30.0000	30.0000
2	-15.0000	-15.0000	-15.0000	-15.0000	-15.0000
3	15.0000	15.0000	15.0000	15.0000	15.0000
4	15.0000	15.0000	15.0000	15.0000	15.0000
5	15.0000	15.0000	15.0000	15.0000	15.0000
6	3.7114	3.8698	3.7120	3.7087	3.7035
7	-3.7114	-3.8698	-3.7120	-3.7087	-3.7035
8	11.2887	11.1302	11.2880	11.2913	11.2965
9	7.327	7.7397	7.339	7.275	7.4070
10	11.2887	11.1302	11.2880	11.2913	11.2965
11	-10.0000	-10.0000	-10.0000	-10.0000	-10.0000
12	1.2886	1.1302	1.2880	1.2913	1.2965
13	7.327	7.7397	7.339	7.275	7.4070
14	1.2886	1.1302	1.2880	1.2913	1.2965
15	10.0000	10.0000	10.0000	10.0000	10.0000
16	1.2886	1.1302	1.2880	1.2913	1.2965
17	-1.2886	-1.1302	-1.2880	-1.2913	-1.2965
18	10.0000	10.0000	10.0000	10.0000	10.0000

[1] EPANET: Gradient Methods implemented in EPANET; FPM: Fixed Point Method; IFPM: Improved Fixed Point Method; IM: Incremental Method; IIM: Improved Incremental Method.

Table 4. Errors on pressures and internal discharges according to the GM.

Errors on pressures	FM	IFPM	IM	IIM
Figure 2.a	0.0260	0.0083	1.2465	0.0121
Figure 2.b	0.0089	0.0077	1.3221	0.0118

Errors on internal discharges	FPM	IFPM	IM	IIM
Figure 2.a	1.4031	0.0030	0.0049	0.0152
Figure 2.b	0.3169	0.0012	0.0052	0.0157

The second example is that from Fig. 2.b, which results are shown in Tab. 3; and the same general conclusions can be draw about the performance of the methods. However, in this case the GM converged after 4 iterations, considering again and accuracy of 10^{-6}.

Note that the errors presented Tab. 4 does not necessarily mean that the alternative methods here presented are less accurate than the GM. These errors can result from small changes in the parameters used in the analysis, like water density, water viscosity or material relative roughness. In fact, the IFPM (which can be considered the most efficient method proposed in this paper) converged after 7 iterations for the example of Fig. 2.a and after 6 iterations for the example of Fig. 2.b, considering an accuracy of 10^{-6}. These values are close to the ones obtained by the GM, which converged after 6 and 4 iterations for the examples from Fig. 2.a and Fig. 2.b respectively.

5. CONCLUSIONS

For the examples presented in this paper, the IFPM and the IIM presented good agreement with the results obtained with the GM. However, the IFPM is currently more adequate for practical use than the IIM, since it can be used for networks with and arbitrary number of prescribed pressures. The IIM, instead, allows only the definition of one prescribed pressure. Besides, the IFPM presented a better convergence rate than the IIM, and is easier to translate into computational routines. However, this picture may change if more effort is applied to the development of the IIM, mainly to the corrector stage. Note that the incremental approach is successfully applied to many structural problems.

Some important features are not discussed in this paper, as the inclusion of pumps and valves. The inclusion of these devices can be accomplished by defining “pipe” elements with appropriate head loss coefficients. Pumps, for example, may be included by defining a “pipe” which gives a head gain, instead of a head loss. However, the inclusion of these and other features (i.e. leakages and nodal discharges dependent on nodal pressures) remain as open questions for future works.

6. REFERENCES

Bathe, K.J, 1996, “Finite Element Procedures”, Prentice Hall Inc., New Jersey, 1037p.
Cross, H., 1936, “Analysis of Flow in Networks of Conduitsor Conductors”, Enginnering Experimental Station: University of Illinois, Bulletin Nº 286.
Formiga, K.T.M, 2005, “Otimização multiobjetivo de projeto de redes de distribuição de água”, PhD. Thesis, Escola de Engenharia de São Carlos, Universidade de São Paulo.
Formiga, K.T.M, Chaudhry, F.H., 2008, “Modelos de análise hidráulica de redes de distribuição de água considerando demanda dirigida pela pressão e vazamentos”, Engenharia sanitária e ambiental, Vol. 13, No. 2, pp. 153-162.
Kutas, R., Čiupailaitë, E, 1997, “The Finite Element Approach for the Analysis of Flow in Pipe Networks”, XIII Polish Conference on Computer Methods in Mechanics, Poznań, Poland.
Larock, B. E., Jeppson, R. W., Watters, G.Z, 2000, “Hydraulics of Pipeline Systems”, CRC Press, New York, 533p.
Nielsen, H.B, 1989, “Methods for Analyzing Pipe Networks”, Journal of Hydraulic Engineering, Vol. 115, No. 2, pp. 139-157.
Roberson, J. A., Cassidy J. J., Chaudhry, M. H, 1998, “Hydraulic Engineering”, John Wiley & Sons, New York, 653p.
Rossman, L.A., 2000, “EPANET 2 User Manual”, Water Supply and Water Resources Division, Enviromental Protection Agency, Cincinnati, 200p.
Simo, J.C., Hughes, T.J.R, 1998, “Computational Inelasticity”, Interdisciplinary Applied Mathematics Series, Vol. 7 Springer-Verlag, New York, 392p.
Todini, E., 2003, “A more realistic to the “extended period simulation” of water distribution networks”. In: C. Maksimovic, D. Butler, et al (Ed.). Advances in Water Supply Management. Lisse A. A. Balkema Publishers, p.732.
Todini, E., Pilati, S, 1987, "A gradient method for the analysis of pipe networks", International Conference on Computer Applications for Water Supply and Distribution, Leicester Polytechnic, UK.
White, F.M, 2006, “Fluid Mechanics”, McGraw Hill, New York, 826p.

7. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.