A SHORT NOTE ON A LINEAR OPTIMAL CONTROL APPLIED TO A VIBRATING SYSTEM, TAKEN INTO ACCOUNT AN ENERGY PUMPING PHENOMENON

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Abstract. *In this work, the nonlinear dynamics behavior of a two-degree-of-freedom system of two coupled and damped oscillators, it is investigated. We proposed the use of an optimal linear feedback control strategy, applied to the adopted mathematical model . This kind control strategy reduced both the movement of this system to a stable orbit and the energy consumption expended by the dynamical system.*

Keywords: *Energy Pumping; Linear Optimal Control; Nonlinear Vibration System*

1. INTRODUCTION

The energy pumping concept has been introduced by Vakakis and Gendelman (Gendelman, 2001; Vakakis, 2001; Vakakis, Gendelman, 2001) in 2001: the main idea is to passively reduce the vibrations in a linear system (discrete or continues) by attaching to it an essentially nonlinear damped oscillator.

The energy pumping phenomenon corresponds to a controlled one-way channeling of the vibrational energy to a passive nonlinear structure, where it localizes and diminishes in time, due to the damping dissipation (Gendelman, 2001; Vakakis, 2001; Vakakis, Gendelman, 2001; Gendelman *et al*, 2001).

So, nonlinear energy pumping can be used in coupled mechanical oscillators (Gendelman, 2003) where the essential nonlinearity of the attachment enables it to resonate with any of the linearized modes of the substructure (Vakakis *et al*, 2003).

We also remarked that the concept of extracting energy away from a system in a simple fashion, so as to reduce its amplitude of vibration, is a novel phenomenon and it forms the basis of concept of energy pumping. Malatkar and Nayfeh (Malatkar, Nayfeh, 2007) investigated the effectiveness of the nonlinear energy sink in vibration attenuation of a linear structure under harmonic excitation. Jiang *et al*. (Jiang *et al*, 2003) studied theoretically and experimentally the forced state dynamics of a system of coupled oscillators composed of a linear subsystem with a nonlinear energy sink attached. They showed the steady state energy pumping in the frequency domain, and that the nonlinear energy sink is capable of absorbing energy from the linear subsystem over a broad frequency range. Vakakis (Vakakis, 2001) studied the energy pumping in a two-degree-of-freedom system as well as in a multi-DOF chain with a weakly coupled nonlinear attachment. He showed that after some initial transients, the response of the nonlinear attachment settles in a motion dominated by a “fast” frequency identical to the lower bound of the propagation zone of the linear chain.

An experimental verification of the theoretical effects of energy pumping especially with external excitation was done in (Gourdon *et al*, 2007). Dynamic responses of a linear oscillator coupled to a nonlinear energy sink (NES), under harmonic forcing in the regime of 1:1:1 resonance was investigated in (Starosvetsky, Gendelman, 2008), and the suppression of vibrations in a 2dof forced linear system with a nonlinear energy sink (NES) attached was reported in (Starosvetsky, Gendelman, 2008).

Finally, we mention that, using a formal mathematical tools and a numerical experiments, some extensions of the nonlinear dynamics of the system those were studied before were explored by Dantas and Balthazar (Dantas, Balthazar, 2008; Costa *et al*, 2008) and (Felix *et al*, 2008).

Recently, a technique proposed by Rafikov and Balthazar in (Rafikov, Balthazar, 2005, 2008): the Dynamic Programming was used to solve the formulated optimal control problems. The linear feedback control problem for nonlinear systems has been formulated under optimal control theory viewpoint. Asymptotic stability of the closed-loop nonlinear system is guaranteed by means of a Lyapunov function, which can clearly be seen to be the solution of the Hamiton-Jacobi-Bellman equation thus guaranteeing both stability and optimality. The formulated theorem expresses explicitly the form of minimized functional and gives the sufficient conditions that allow using the linear feedback control for nonlinear system. This technique was used in another kind of problems by (Chavarette *et al*, 2009a; 2009b; 2009c; Pereira *et al*, 2007).

The aim of this paper is to discuss an application of this referenced optimal linear control theory of linear structure by coupling it to adapted passive nonlinear structure. Noted that this kind of control was used before, with success, by (Costa and Balthazar, 2009) in the control the level of energy pumping by choosing suitably values of the physical parameters given in (Gourdon and Lamarque, 2005a, 2005b). To do it, we will organize this paper, as follows: Section 2 begins with the two-degree-of-freedom system with two coupled and damped oscillators. Section 3, describes the application of the optimal linear control in the considered mathematical models. In section 4, we do some concluding remarks of this work and in section 5, we do some acknowledgements. Then we list the main bibliographic references used.

2. MATHEMATICAL MODEL: THE MECHANICAL SYSTEM OVERVIEW

We deal with an extension of the nonlinear dynamics of the system studied before in an ideal case, by (Gordon, Lamarque, 2005a, 2005b). It consists of a linear structure coupled to a nonlinear energy sink. A schematic of this coupled dynamical system in shown in Figure 1, where denote the M, k_2, c_2, $x_2(t)$ unbalanced mass, mass, linear stiffness, linear damping, displacement and m_0, m, k_1, c_1, $x_1(t)$ denote the mass, nonlinear stiffness, linear damping, displacement of the essentially nonlinear oscillator; γ denotes the weak linear coupling stiffness between the subsystems and t is time. Structures may be represented by systems mass-spring-damper using model's reduction techniques, as shown in figure 1.

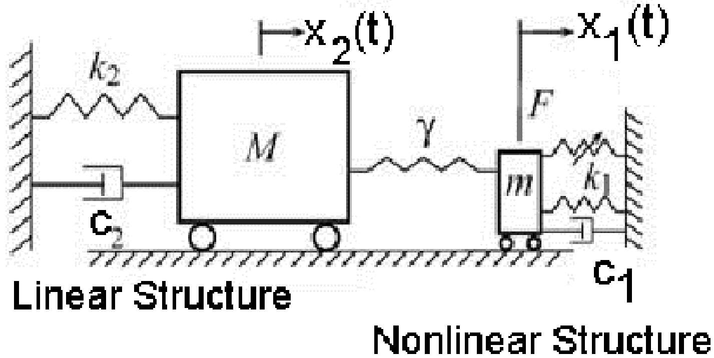

Figure 1. General system with the two-degree-of-freedom (Gourdon, Lamarque, 2005a, 2005b).

The governing equations of motion of the considered vibrating system, defined by Figure 1, are

$$\begin{cases} m_1\ddot{x}_1 + c_1\dot{x}_1 + k_1x_1 + k_2(x_1 - x_2) = F \\ m_2\ddot{x}_2 + c_2\dot{x}_2 + k_2(x_2 - x_1) = 0 \end{cases} \tag{1}$$

Here, x_1 and x_2 denote displacements of the absorber and main linear systems respectively and the mass is m, k is spring, c is damper and $F=F_0cos(wt)$.

The non-dimensional equation (1) it is (Gourdon, Lamarque, 2005a, 2005b):

$$\left.\begin{array}{l}\tau = w_1 t, \quad w_1 = \sqrt{\dfrac{k_1}{m_1}}, \quad w_2 = \sqrt{\dfrac{k_2}{m_2}}, \\ x_1 = \dfrac{X}{l_1}, \quad x_2 = \dfrac{Y}{l_1}, \; \Omega = \dfrac{w}{w_1},. \\ \dot{X}_1 = X_1' w_1, \quad \ddot{X}_1 = w_1^2 X_1', \\ \dot{X}_1 = w_1 l X', \quad \ddot{X}_1 = w_1^2 l_1 X' \\ \alpha_1 = \dfrac{c_1}{m_1 w_1}, \quad \alpha_2 = \dfrac{k_2}{m_1 w_1^2}, \quad \alpha_3 = \dfrac{c_2}{m_2 w_1}, \\ \alpha_2 = \dfrac{k_2}{m_2 w_1^2}, \quad \mathrm{A}_0 = \dfrac{F_0}{w_1^2 m_1}\end{array}\right\} \tag{2}$$

Therefore, equation (1) and (2) give

$$\begin{array}{l}X'' = -\alpha_1 X' - X - \alpha_2 (X - Y) + A_0 \cos(\Omega t) \\ Y'' = -\alpha_3 X' - \alpha_4 (Y - X)\end{array} \tag{3}$$

Rewriting the dynamical system, in the state form, through the transformations

$x_1 = X$, $x_2 = \dot{X}$, $x_3 = Y$ and $x_4 = \dot{Y}$

we will obtain:

$$\begin{array}{l}\dot{x}_1 = x_2 \\ \dot{x}_2 = -\alpha_1 x_2 - x_1 - \alpha_2 (x_1 - x_3) + A_0 \cos(\Omega t) \\ \dot{x}_3 = x_4 \\ \dot{x}_4 = -\alpha_3 x_3 - \alpha_4 (x_3 - x_1)\end{array} \tag{4}$$

The Figures 2 illustrate the behavior of the dynamics model, by using numerical values, for the parameters α_1=0.001; α_2=0.09; α_3=0.9; α_4=0.2; A_0=0.95, Ω=3.1415; x_1=0.2; x_2=0; x_3=1 and x_4=1.

Figure 2a presents the time history of x1 and x_3, in Figure 2b shows the time history of x2 and x_4, in Figure 2c the projection of the phase (portrait) space for first oscillator it is illustrated; in Figure 2d the projection of the phase space for the second oscillator it is illustrated; in Figure 2e the diagram of the stability (with the region's control applied it is illustrated.)

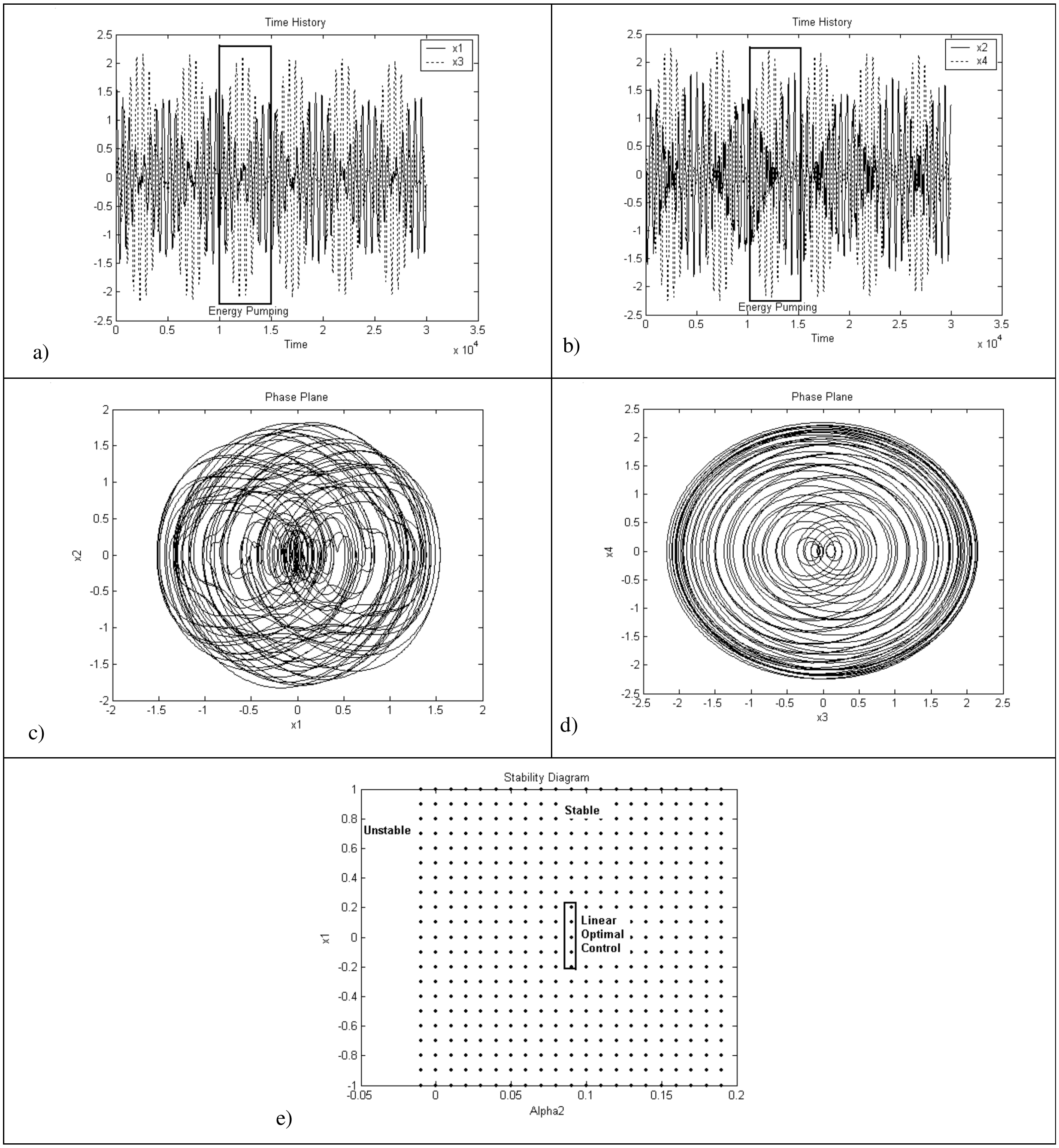

Figure 2. (a) . The behavior of the time history x_1 and x_3.; (b The behavior of the time history x_2 and x_4.; (c) The projection of the phase space (x_1,x_2) to the first oscillator; (d) The projection of the phase space (x_3, x_4) to the second oscillator, and (e) Stability Diagram with the region's control applied.

Energy pumping corresponds to control of linear structure by coupling it to adapted passive nonlinear structure (Gendelman, 2001; Vakakis, 2001; Vakakis, Gendelman, 2001; Gendelman, *et al*, 2001) as shown in Figure 2. However, as described in (Vakakis, 2001), energy pumping occurs above a specific value of the initial energy level: so when energy injected is too low, energy transfer from the linear structure to the nonlinear one does not appear. When energy pumping occurs, energy decreases in the linear structure first due to the transient nonlinear efficient phenomenon, then only due to damping dissipation (less efficient) when residual energy is too low in the coupled system.

Figure 3 presents the behavior of x_1, x_2, x_3 and x_4 with F=0 proposed by (Costa *et al*, 2008) and F=Fo*cos(wt) adopted in this work. We note that F (external excitation) applied to the system increases its amplitude.

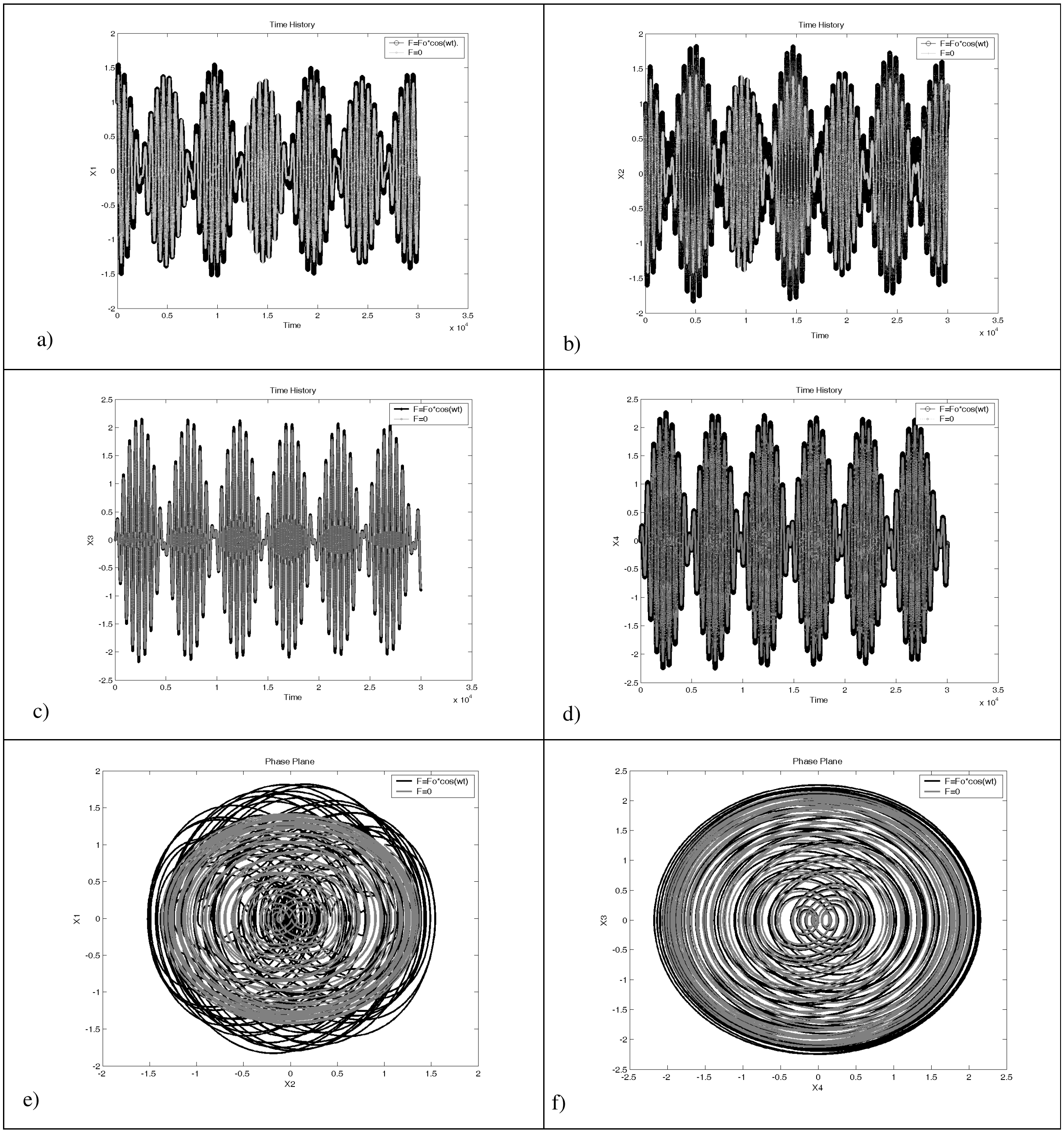

Figure 3. (a) . The behavior of the time history x_1 (F=Fo*cos(wt);'F=0); (b The behavior of the time history x_2 (F=Fo*cos(wt);'F=0); (c) The behavior of the time history x_3 (F=Fo*cos(wt);'F=0); (d) The behavior of the time history x_4 (F=Fo*cos(wt);'F=0); (e) The projection of the phase space (x_1,x_2) to the first oscillator (F=Fo*cos(wt);'F=0); (f) The projection of the phase space (x_3, x_4) to the second oscillator (F=Fo*cos(wt);'F=0).

Here, we applied optimal linear control design to reduce the movement of this system to a stable orbit.

3. LINEAR OPTIMAL CONTROL PROJECT (Rafikov and Balthazar 2005, 2008)

In this section, we applied optimal linear control design, for the two-degree-of-freedom system composed of two coupled and damped oscillators (figure 1), reducing the oscillatory movement to a stable orbit.

Next, we present the theory of the used methodology.

Due to the simplicity in configuration and implementation, the linear state feedback control, it is especially attractive (Rafikov, Balthazar, 2005; 2008; Chavarette *et al*, 2009a; 2009b).

We remarked that this approach is analytical, and it may used without dropping any non-linear term.

Let the governing equations of motion (4), re-written in a state form

$$\dot{x} = Ax + g(x) \tag{5}$$

If one considers a vector function $\tilde{x}$, that characterizes the desired trajectory, and taken the control U vector consisting of two parts: $\tilde{u}$ being the feed forward and u_f is a linear feedback, in such way that

$$u_t = Bu \tag{6}$$

where B is a constant matrix. Next, one taking the deviation of the trajectory of system (5) to the desired one (7) $y = x - \tilde{x}$, may written as being

$$\dot{y} = Ay + g(x) - g(\tilde{x}) + Bu \tag{7}$$

where $G(y, \tilde{x})$ is limited matrix we proved the important result (Rafikov, Balthazar, 2005; 2008).

If there exit matrices *Q(t)* and *R(t)*, positive definite, being *Q* symmetric, such that the matrix $\tilde{Q} = Q - G^T(y, \tilde{x})P(t) - P(t)G(y, \tilde{x})$ is positive definite for the limited matrix *G*, then the linear feedback control

$$u = -R^{-1}B^T P\, y \tag{8}$$

It is optimal, in order to transfer the non-linear system (6) from any initial to final state $y(t_f)=0$, minimizing the functional $\tilde{J} = \int_0^\infty (y^T \tilde{Q}\, y + u^T R u)dt$, where the symmetric matrix *P(t)* is evaluated through the solution of the matrix Ricatti differential equation

$$PA + A^T P - PBR^{-1}B^T P + Q = 0 \tag{9}$$

Satisfying the final condition $P(t_f)=0$.

In addition, with the feedback control (9), there exists a neighborhood $\Gamma_0 \subset \Gamma$, $\Gamma \subset \Re^n$, of the origin such that if $x_0 \in \Gamma_0$, the solution $x(t) = 0,\ t \geq 0,$ of the controlled system (7) is locally asymptotically stable, and $J_{\min} = x_0^T P(0)\, x_0$. Finally, if $\Gamma = \Re^n$ then the solution $y(t) = 0,\ t > 0,$ of the controlled system (7) is globally asymptotically stable.

3.1 UNCONTROLED AND CONTROLLED VIBRATING SYSTEM

To the application of the technique of controlling the system, the equations Eq. (4), describing the model controlled, may be rewritten in the following form

$$\begin{aligned} \dot{x}_1 &= x_2 + U \\ \dot{x}_2 &= -\alpha_1 x_2 - x_1 - \alpha_2(x_1 - x_3) + A_0\cos(\Omega t) \\ \dot{x}_3 &= x_4 \\ \dot{x}_4 &= -\alpha_3 x_3 - \alpha_4(x_3 - x_1) \end{aligned} \tag{10}$$

Then, we will obtain

$$B = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad y = \begin{bmatrix} x_1 - \tilde{x}_1 \\ x_2 - \tilde{x}_2 \\ x_3 - \tilde{x}_3 \\ x_4 - \tilde{x}_4 \end{bmatrix}, \quad \tilde{x} = \begin{bmatrix} 0.2 \\ 0.4 \\ 0 \\ 0 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1.09 & -0.001 & 0.09 & 0 \\ 0 & 0 & 0 & 1 \\ 0.2 & 0 & -1.1 & 0 \end{bmatrix}$$

where $M\left|B \mid AB \mid ABC \mid ABCD\right| \neq 0$ (then the considered dynamical system it is controllable).

Then the Matrix *P(t)* is done by $P = \begin{bmatrix} 22.9798 & 0.3969 & -22.2714 & 0.9176 \\ 0.3969 & 18.7721 & -1.7221 & -17.2828 \\ -22.2714 & -1.7221 & 23.4554 & -0.3236 \\ 0.9176 & -17.2828 & -0.3236 & 19.2696 \end{bmatrix}$ and (an optimal control) $u = 0.7084x_1 - 1.32522x_2 + 1.1840x_3 + 0.5940x_4$. The trajectories of the system, without control and with control may be seen through Figure 4.

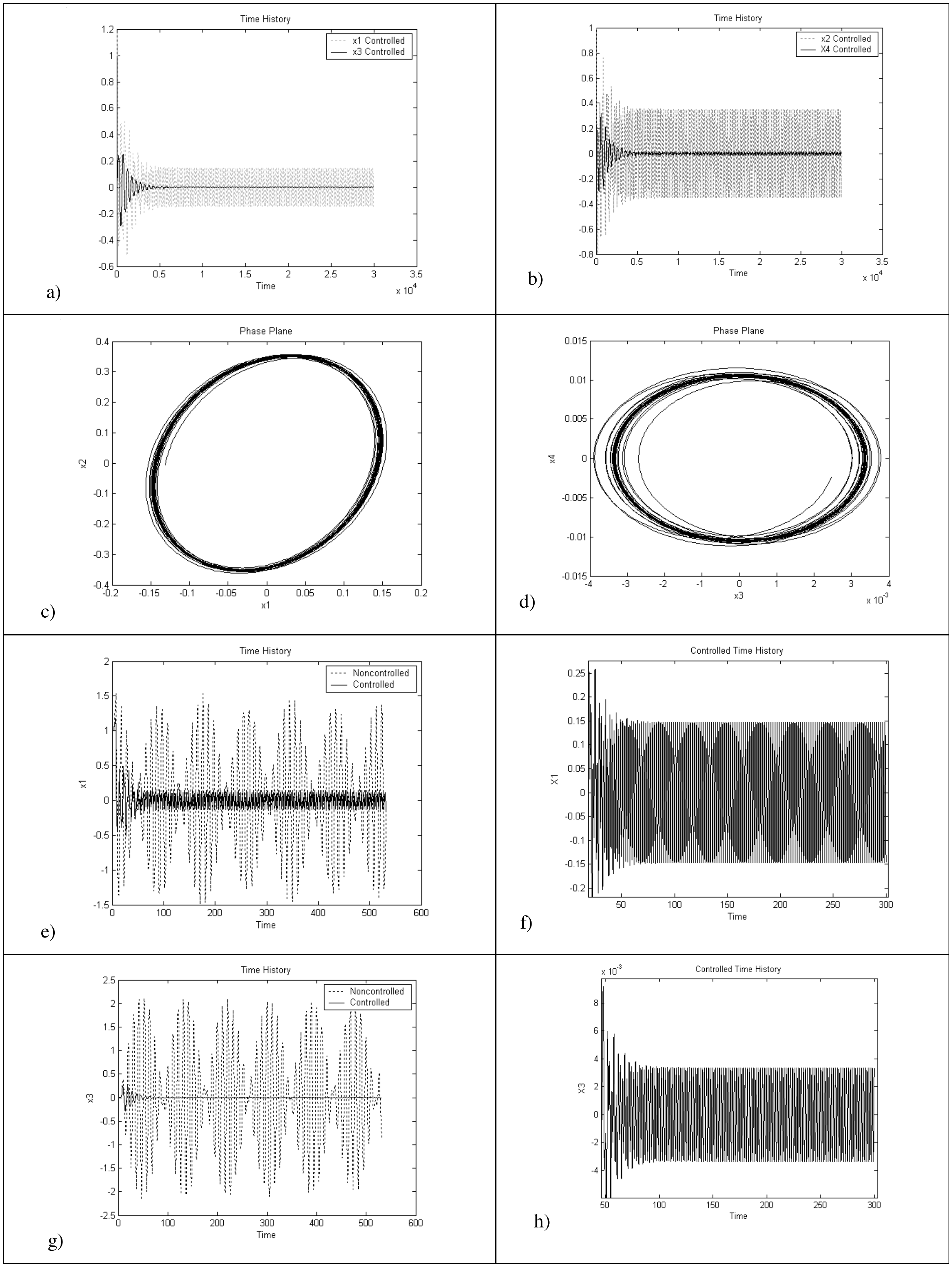

Figure 4. (a) The behavior controlled of the time history x_1 and x_3; (b) The behavior controlled of the time history x_2 and x_4.; (c) The projection of the controlled phase space (x_1,x_2) to the first oscillator; (d) The projection of the phase space (x_3, x_4) to the second oscillator, (e) The behavior controlled and non-controlled of the time history x_1; (f) The behavior controlled of the time history x_1 ; (g) The behavior controlled and non-controlled of the time history x_3, and (h) The behavior controlled of the time history x_3.

The Figure 5a illustrates the controlled and non-controlled energy of both systems with *F=Fo*cos(wt),* the Figure 5b illustrates the controlled energy of both systems with *F=Fo*cos(wt)*, the Figure 5d illustrates the controlled and non-controlled energy of both systems with *F=0* and*,* the Figure 5d illustrates the controlled energy of both systems with *F=0.*

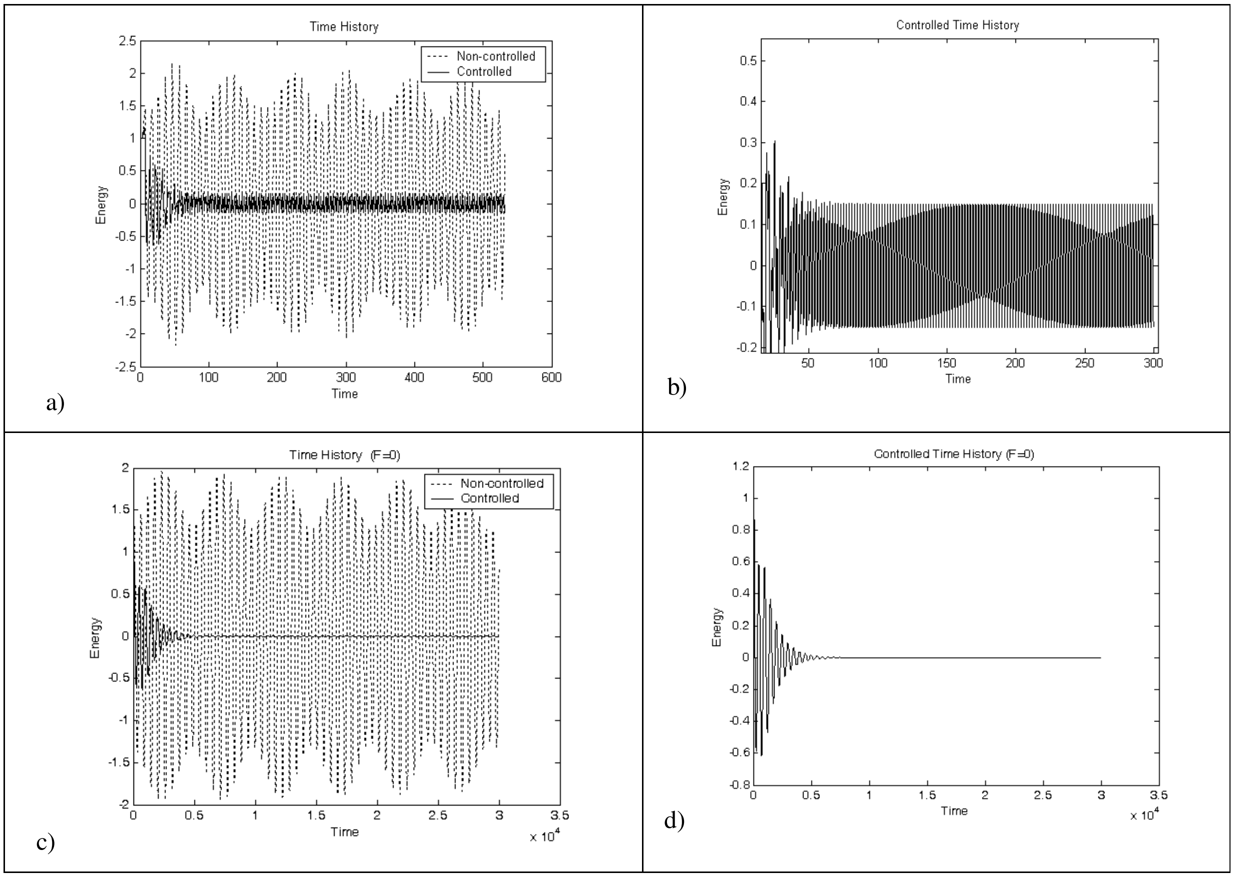

Figure 5. (a) The behavior controlled and non-controlled energy of both in the system (F=Fo*cos(wt)); (b) The behavior controlled energy of both in the system (F=Fo*cos(wt)); (c) The behavior controlled and non-controlled energy of both in the system (F=0), and (d) The behavior controlled energy of both in the system (F=0);

4. CONCLUSION

In this work, the dynamics of the two-degree-of-freedom system, with two coupled and damped oscillators it is investigated by numerical simulations. The preliminary sets of numerical simulations carried out, clearly showed that the main objective of this paper was reached.

We proposed the use of an optimal linear feedback control strategy, applied to the mathematical model proposed for reducing the vibration amplitude of the considered system, according to the experimental work of Jiang *et al* (Jiang *et al*, 2003). This kind control strategy reduced the movement of this system to a stable orbit. The Figure 4 and 5 illustrated the effectiveness of the control strategy to these problems.

On the other hand, Figure 5 illustrate that the energy consumption in the system is greater than to the than to control system, showing that the optimal linear control applied to the system reducing the energy consumption expended by the system. Figure 5a shows amplitude greater than the figure 5e due to external excitation *(F)*. Figure 5c and 5d shows that the system without external excitation *(F=0)* the controller stabilizes the system to the origin, unlike the figures 5a and 5b.

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