CONSISTENT DENSITY-CORRECTED TANGENT OPERATORS FOR DENSITY-DEPENDENT MATERIALS - ANALYTICAL DERIVATION

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Abstract. *In this work an analytical strategy to derive consistent density-corrected tangent operators for compressible materials in the context of finite deformation is proposed. It is shown that the density dependence of some model variables implies corrections in the tangent operators. To derive this called density-corrected tangent operators an algorithm consistent equation is devised and added to the set of return mapping nonlinear equations only at the moment of linearization. Examples of such corrections are presented considering two different elastoplastic compressible models inside the context of finite deformations. Numerical examples showing the influence of the density-correction term in the global convergence are presented in order to attest the proposed methodology.*

Keywords: *Tangent operators, Powder compaction, Geomechanical materials, Density-dependent materials.*

1. INTRODUCTION

As well stated by Doraivelu *et. al.* (1984) "*In the development of plasticity models for compressible porous materials it is necessary to establish a yield criterion and a flow rule from which the complementary evolution equations can be derived. However, the yielding of porous materials is much more complex than the yielding of fully dense materials, mainly due to the fact that the yielding is not only influenced by the deviatoric part of the stress, but also by its hydrostatic part*".

Examples of compressible materials are soils, powders and foams. Each one of these materials has its particularity, which influences its modeling. In the literature, it is possible find many original models and dozens of their variations. In general soils are modeled by *Cap* models, powders are modeled also by *Cap* models and by the so called elliptical models, a variation of the *von Mises* criterion that incorporates the hydrostatic part of the stress, and foams are in general modeled by the usage of a specific model that takes into account the foam microstructure.

To successfully model compressible porous materials by means of computational efficiency, *CPU* time consuming and convergence rates, it is necessary, despite the correct material model choice, to derive the so called consistent or continuum tangent operator. A particular characteristic of compressible porous materials is that some model constitutive variables can depend on the density evolution. In a first look such a dependence do not bring any important novelty once the solution of the so called return mapping equations takes place by assuming a known fixed value of the displacement vector and therefore a fixed density.

However, to achieve better convergence rates it is imperative that during the linearization of the return mapping equations one takes into account the density as variable. Therefore an equation comprising the relationship between the density and the strain, for instance, have to be identified. This equation shall be consistent with the description and algorithm in use. Pérez-Foguet *et al.* (2001,2003) presented how to derive consistent tangent matrices for density-dependent finite plasticity models inside the arbitrary Lagrangian-Eulerian description. A general equation for the tangent operator was presented, Pérez-Foguet *et al.* (2001), based on the previously density-independent results presented by Ortiz & Martin (2000). The tangent operators for the elastoplastic models presented in Pérez-Foguet *et al.* (2001,2003) were then computed by using numerical strategies presented by Pérez-Foguet *et. al.* (2000).

In this work an analytical strategy to derive consistent density-corrected tangent operators for compressible materials is derived. The proposed procedure assumes the *Total Lagrangian* description and considers: a multiplicative decomposition of the deformation gradient, into a plastic and an elastic part; the isotropic constitutive formulation given in terms of the logarithmic deformation measure and the rotated *Kirchhoff* stress so that the exponential return mapping can be used. In this work the elastic response is assumed to be hyperelastic, according to the *Hencky* model, where the elastic material parameters can or can not depend on the relative density.

2. DENSITY-DEPENDENT FINITE PLASTICITY MODELS

Inside the context of the mathematical theory of fully dense materials, density-independent materials, it is stated that the functional form of the yield function f has dependence on the stress tensor and on a set of β hardening thermodynamical forces.

Despite the functional form of the mathematical model used in the modeling of density-dependent finite plasticity models they usually have some parameters that are dependent on the evolution of the mass density ρ

$$\rho = \frac{\rho_o}{\det(\mathbf{F})}, \tag{1}$$

where ρ_o is the initial mass density and $\mathbf{F}$ is the deformation gradient. This equation can be presented in terms of relative density η which is defined as

$$\eta = \frac{\rho}{\rho_m} = \frac{\eta_o}{\det(\mathbf{F})}, \tag{2}$$

in which ρ_m is the mass density of fully dense material and $\eta_o = \frac{\rho_o}{\rho_m}$. Therefore, in the modeling of density-dependent finite plasticity the yield function shall be written as

$$f = f(\bar{\tau}, \beta, \eta), \tag{3}$$

in which$\bar{\tau}$ is the rotated *Kirchhoff* stress. In addition to the yield criteria an elastoplastic constitutive model has to establish the evolution of the plastic flow rule and of the internal variables. Here the constitutive elastoplastic model can by summarized, by assuming an associative flow[1] rule, as to enforce at every point where $f > 0$ the satisfaction of the following set of nonlinear equations

$$f(\bar{\tau}, \beta, \eta) = 0 \tag{4a}$$

$$\bar{\mathbf{D}}^p = \dot{\lambda}\frac{\partial f}{\partial \bar{\tau}} = \dot{\lambda}\mathbf{N}_{\bar{\tau}} \tag{4b}$$

$$\dot{\alpha} = \dot{\lambda}\frac{\partial f}{\partial \beta} = \dot{\lambda}\mathbf{N}_{\beta} \tag{4c}$$

In equation Eq.(4b) and Eq.(4c) $\dot{\lambda}$ is the plastic multiplier, which is determined by the satisfaction of the *Karush-Kuhn-Tucker* conditions

$$f \leq 0 \qquad \dot{\lambda} \geq 0 \qquad \dot{\lambda} f = 0, \tag{5}$$

$\bar{\mathbf{D}}^p$ is the modified plastic evolution, $\dot{\alpha}$ plays the role of the internal variables evolution vector, β denotes the vector of internal variables which are associated with α. Besides the constitutive equation for the stress is

$$\bar{\tau} = \mathbb{D}^e(\eta)\,\mathbf{E}^e \tag{6}$$

where $\mathbb{D}^e$ is the fourth order elasticity tensor, $\mathbf{E}^e = \ln(\mathbf{U}^e)$ is the *Hencky* strain measure, $\mathbf{U}^e = \sqrt{\mathbf{C}^e}$, $\mathbf{C}^e = (\mathbf{F}^e)^T \mathbf{F}^e$ and $\mathbf{F} = \mathbf{F}^e\mathbf{F}^p$.

3. THE MATERIAL TANGENT MODULUS

It is well known that the material version of the tangent modulus, assuming the finite strain case, can be derived if one assumes the first *Piola-Kirchhoff* stress tensor as a function of the deformation gradient

$$\mathbf{P} = \mathbf{P}(\mathbf{F}(\mathbf{u})) \tag{7}$$

in which $\mathbf{u}$ is the displacement vector. As a result the material version of the virtual work is given by

$$\mathcal{G}(\mathbf{u}, \hat{\mathbf{u}}) = 0 \qquad \forall\ \hat{\mathbf{u}} \in \mathcal{W}_p^1(\Omega_o) \tag{8}$$

with

$$\mathcal{G}(\mathbf{u}, \hat{\mathbf{u}}) = \int_{\Omega_o} \mathbf{P}(\mathbf{u}) \cdot \nabla_{\mathbf{X}}\hat{\mathbf{u}}\, d\Omega_o - \int_{\Omega_o} \rho_o\bar{\mathbf{b}} \cdot \hat{\mathbf{u}}\, d\Omega_o - \int_{\Gamma_o^t} \bar{\mathbf{t}} \cdot \hat{\mathbf{u}}\, d\Gamma_o^t \tag{9}$$

where $\bar{\mathbf{b}}$ is the body force field defined in the initial body configuration Ω_o, which has boundary $\partial\Omega_o = \Gamma_o^t \cup \Gamma_o^u$ with $\Gamma_o^t \cap \Gamma_o^u = \emptyset$, $\bar{\mathbf{t}}$ is the prescribed traction defined on boundary Γ_o^t, $\nabla_{\mathbf{X}}(\bullet)$ means the material gradient of the field $(\bullet)$, $\bar{\mathbf{u}}$ is the prescribed displacement defined on Γ_o^u and $\hat{\mathbf{u}}$ is the virtual displacement vector field. The linearization of the weak form presented in Eq.(8)

$$D\mathcal{G}(\mathbf{u}, \hat{\mathbf{u}})[\delta\mathbf{u}] = \frac{d}{d\epsilon}\, \mathcal{G}(\mathbf{u} + \epsilon\delta\mathbf{u}, \hat{\mathbf{u}})\big|_{\epsilon=0} \tag{10}$$

[1]Non associative functions could also be used.

leads to the linearization of the term associated with the internal virtual work, i.e.,

$$\frac{d}{d\epsilon}\int_{\Omega_o}\mathbf{P}\left(\mathbf{u}\right)\cdot\nabla_{\mathbf{X}}\hat{\mathbf{u}}\,d\Omega_o\bigg|_{\epsilon=0}$$

which can be expressed, after a straightforward algebric manipulation, as

$$D\mathcal{G}\left(\mathbf{u},\hat{\mathbf{u}}\right)\left[\delta\mathbf{u}\right]=\int_{\Omega_o}\mathbb{A}\left(\mathbf{u}\right)\nabla_{\mathbf{X}}\mathbf{u}\cdot\nabla_{\mathbf{X}}\hat{\mathbf{u}}\,d\Omega_o \tag{11}$$

where$\mathbb{A}$ is fourth order tensor called *material tangent modulus* that can be written as

$$\mathbb{A}\left(\mathbf{u}\right)=\left.\frac{\partial\mathbf{P}}{\partial\mathbf{F}}\right|_{\mathbf{u}} \tag{12}$$

or more specifically in components

$$A_{ijkl}=\frac{\partial P_{ij}}{\partial F_{kl}}=\frac{\partial\tau_{ip}}{\partial F_{kl}}F_{jp}^{-1}-\tau_{ip}F_{jk}^{-1}F_{lp}^{-1}. \tag{13}$$

Here τ is the *Kirchhoff* stress tensor, $\tau=J\sigma$ with σ denoting the *Cauchy* stress tensor and $J=\det\left(\mathbf{F}\right)$. τ is related to the first *Piola-Kirchhoff* stress tensor by $\mathbf{P}=\tau\mathbf{F}^{-T}$.

4. ALGORITHM CONSIDERATIONS AND TANGENT OPERATOR CORRECTIONS

4.1 Determination of the tangent modulus $\mathbb{A}$

The determination of the tangent modulus $\mathbb{A}$ requires the computation of the derivative of the *Kirchhoff* stress tensor with relation to the deformation gradient tensor, as seem in Eq.(12). In addition, it is possible to relate *Kirchhoff* stress tensor with the rotated *Kirchhoff* stress tensor and therefore write τ as a function of the $\overline{\tau}$. In other words, the computation of $\frac{\partial\tau}{\partial\mathbf{F}}$ requires the computation of $\frac{\partial\overline{\tau}}{\partial\mathbf{F}}$.

Taking into account the incremental constitutive formulation, see Souza Neto *et al.* (1998), the derivative of the rotated *Kirchhoff* stress tensor with relation to the deformation gradient tensor can be evaluated realizing that

$$\overline{\tau}_{n+1}=\hat{\overline{\tau}}_{n+1}\left(\mathbf{E}_{n+1}^{e^{trial}},\beta_n\right). \tag{14}$$

Now, by applying the chain rule

$$\frac{\partial\overline{\tau}_{n+1}}{\partial\mathbf{F}_{n+1}}=\frac{\partial\overline{\tau}_{n+1}}{\partial\mathbf{E}_{n+1}^{e^{trial}}}\frac{\partial\mathbf{E}_{n+1}^{e^{trial}}}{\partial\mathbf{C}_{n+1}^{e^{trial}}}\frac{\partial\mathbf{C}_{n+1}^{e^{trial}}}{\partial\mathbf{F}_{n+1}} \tag{15}$$

and denoting

$$\mathbb{D}=\frac{\partial\overline{\tau}_{n+1}}{\partial\mathbf{E}_{n+1}^{e^{trial}}},\ \mathbb{G}=\frac{\partial\mathbf{E}_{n+1}^{e^{trial}}}{\partial\mathbf{C}_{n+1}^{e^{trial}}}\text{ and }\mathbb{H}=\frac{\partial\mathbf{C}_{n+1}^{e^{trial}}}{\partial\mathbf{F}_{n+1}} \tag{16}$$

it is possible rewrite the Eq.(15) as

$$\frac{\partial\overline{\tau}_{n+1}}{\partial\mathbf{F}_{n+1}}=\mathbb{DGH}. \tag{17}$$

Starting from $\mathbf{C}_{n+1}^{e^{trial}}=\left(\mathbf{F}_{n+1}^{e^{trial}}\right)^T\mathbf{F}_{n+1}^{e^{trial}}$ the $\mathbb{H}$ components, after a straightforward algebric manipulation, can be expressed as

$$H_{ijkl}=F_{li_n}^{p^{-1}}F_{kj_{n+1}}^{e^{trial}}+F_{ki_{n+1}}^{e^{trial}}F_{lj_n}^{p^{-1}}. \tag{18}$$

The fourth order tensor $\mathbb{G}$ is computed by the following expression

$$\mathbb{G}=\frac{\partial}{\partial\mathbf{C}_{n+1}^{e^{trial}}}\ln\left(\mathbf{U}_{n+1}^{e^{trial}}\right)=\frac{1}{2}\frac{\partial}{\partial\mathbf{C}_{n+1}^{e^{trial}}}\ln\left(\mathbf{C}_{n+1}^{e^{trial}}\right). \tag{19}$$

Note that in $\mathbb{G}$ determination we need to compute a derivative that involves $\frac{\partial\ln(\mathbf{X})}{\partial\mathbf{X}}$, that is a derivative of the isotropic function $\ln\left(\mathbf{X}\right)$. This class of functions and their derivatives are investigated in details in the works presented by Souza Neto *et al.* (1998) and Ortiz *et al.* (2001). In the Eq.(17) the fourth order tensor $\mathbb{D}$ is the term that involves the material constitutive relationship. The other two are related with geometric portion of the tangent modulus. In fact, the derivation of $\mathbb{D}$ will depend on the type of material being modeled, i.e., in the case of a material that exhibits elastic and inelastic behavior, if the yield function $f\leq0$ then $\mathbb{D}$ is taken as the elastic modulus $\mathbb{D}^e$, otherwise if $f>0$ then $\mathbb{D}$ will be the consistent elastoplastic tangent operator $\mathbb{D}^{ep}$.

4.2 The elastic predictor and plastic corrector algorithm

To understand how the relative density influence arises into tangent operators it is important to review the returning mapping algorithm used in the solution of the *local equations* and how it relates to the determination of the tangent operator $\mathbb{D}^{ep}$. The use of the approximation via the elastic predictor and plastic corrector algorithm technique leads to an algorithm based on two main steps. They are:

1. Elastic prediction: the problem is assumed to be purely elastic between t_n e t_{n+1};
2. Plastic correction: by the enforcement of the elastic relation, plastic flow rule, the evolution of hardening variables (internal variables) and the satisfaction of the *Karush-Kuhn-Tucker* conditions.

4.2.1 Elastic prediction

In the elastic prediction it is assumed that

$$\dot{\mathbf{F}}^p = \mathbf{0} \qquad \dot{\alpha} = \mathbf{0}. \tag{20}$$

As the solution is former assumed as elastic then

$$\mathbf{F}_{n+1}^{p^{trial}} = \mathbf{F}_n^p \qquad \alpha_{n+1}^{trial} = \alpha_n. \tag{21}$$

The called *trial elastic state* is obtained by means of

$$\mathbf{F}_{n+1}^{e^{trial}} = \mathbf{F}_{n+1} \left(\mathbf{F}_n^p\right)^{-1}. \tag{22}$$

This implies that the logarithmic strain measure is computed by

$$\mathbf{E}_{n+1}^{e^{trial}} = \frac{1}{2} \ln \mathbf{C}_{n+1}^{e^{trial}} \tag{23}$$

with $\mathbf{C}_{n+1}^{e^{trial}} = \left(\mathbf{F}_{n+1}^{e^{trial}}\right)^T \mathbf{F}_{n+1}^{e^{trial}}$. Since that $\mathbf{E}_{n+1}^{e^{trial}}$ is determined, then it is possible determine the trial rotated *Kirchhoff* stress tensor by the use of the elastic relation,i.e,

$$\bar{\tau}_{n+1}^{trial} = 2\mu\left(\eta_{n+1}\right) \mathbf{E}_{n+1}^{e^{trial}} + \left[\kappa\left(\eta_{n+1}\right) - \frac{2}{3}\mu\left(\eta_{n+1}\right)\right] tr\left(\mathbf{E}_{n+1}^{e^{trial}}\right) \mathbf{I} \tag{24}$$

where we assume the standard fourth order elasticity tensor being also dependent on the relative density, that means

$$\mathbb{D}^e = 2\mu\left(\eta_{n+1}\right) \mathbb{I} + \left[\kappa\left(\eta_{n+1}\right) - \frac{2}{3}\mu\left(\eta_{n+1}\right)\right] \mathbf{I} \otimes \mathbf{I}. \tag{25}$$

4.2.2 Plastic correction

The plastic correction must be performed if $f\left(\bar{\tau}_{n+1}^{trial}, \alpha_{n+1}^{trial}, \eta_{n+1}\right) > 0$. The procedure adopted to perform the plastic correction belongs to the return mapping algorithms, extensively explored in literature. In this work, as proposed by Eterovic and Bathe (1990) and Weber and Anand (1990), the exponential mapping is used.

Exponential return mapping The discretization of the plastic flow, $\dot{\mathbf{F}}^p = \bar{\mathbf{D}}^p \mathbf{F}^p$, and its approximation based on the backward exponential mapping leads to

$$\mathbf{F}_{n+1}^p = \exp\left(\Delta\lambda \mathbf{N}_{\bar{\tau}_{n+1}}\right) \mathbf{F}_n^p. \tag{26}$$

In addition, the evolution of the internal variables are approximated based on the backward Euler, i.e.,

$$\alpha_{k_{n+1}} = \alpha_{k_n} + \Delta\lambda \mathbf{N}_{\beta_{k_{n+1}}}. \tag{27}$$

Moreover, after a straightforward manipulation, Eq.(26) reduces to

$$\mathbf{E}_{n+1}^e = \mathbf{E}_{n+1}^{e^{trial}} - \Delta\lambda \mathbf{N}_{\bar{\tau}_{n+1}}. \tag{28}$$

Also, it can be shown that $\mathbf{R}_{n+1}^e = \mathbf{R}_{n+1}^{e^{trial}}$. As a result, the return mapping algorithm comprises the solution of the following non-linear system of equations

$$\left\{ \begin{array}{l} \mathbf{E}_{n+1}^e - \mathbf{E}_{n+1}^{e^{trial}} + \Delta\lambda \mathbf{N}_{\bar{\tau}_{n+1}} \\ \alpha_{k_{n+1}} - \alpha_{k_n} - \Delta\lambda \mathbf{N}_{\beta_{k_{n+1}}} \\ f\left(\bar{\tau}_{n+1}, \beta_{k_{n+1}}; \eta_{n+1}\right) \end{array} \right. = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ 0 \end{bmatrix} \tag{29}$$

for $\Delta\lambda$, $\alpha_{k_{n+1}}$ and $\mathbf{E}_{n+1}^e$. **Remark:** Notice that based on a fixed incremental displacement at the instant t_{n+1}, that is $\mathbf{u}_{n+1}$, the deformation gradient $\mathbf{F}_{n+1}$ is computed and so the relative density η_{n+1}. Therefore, the relative density is fixed, not a variable, in the context of the return mapping algorithm, see Pérez-Foguet *et al.* (2001).

4.3 Density-corrected consistent tangent operator determination - $\mathbb{D}^{ep}$

In this section we investigate the influence of the density-dependence models into the derivation of consistent tangent operator. The relative density model dependence impose a correction in the tangent operator. A spatial version of the corrected tangent operator is presented in the work by Pérez-Foguet *et al.* (2001). In fact, in the identification of $\mathbb{D}^{ep}$ the the linearization of the return mapping equations must consider the elastic trial strain also as a variable. This implies that $d\mathbf{E}_{n+1}^{e^{trial}} \rightarrow d\eta_{n+1}$. This means that a coupled relation between $d\mathbf{E}_{n+1}^{e^{trial}}$ and $d\eta_{n+1}$ must be derived. The relation between $d\mathbf{E}_{n+1}^{e^{trial}}$ and $d\eta_{n+1}$ should be consistent with the algorithm used. To identify such relation recall that in the elastic prediction step we state that

$$\mathbf{F}_{n+1} = \mathbf{F}_{n+1}^{e^{trial}} \mathbf{F}_n^p. \tag{30}$$

Based on the trial elastic state assumption and on the Eq.(23) and reminding that

$$\mathbf{C}_{n+1}^{e^{trial}} = \exp\left(2\mathbf{E}_{n+1}^{e^{trial}}\right) \tag{31}$$

it is possible show, after a straightforward algebric manipulation, that

$$\det\left(\mathbf{F}_{n+1}^{e^{trial}}\right) = \exp\left(E_{v_{n+1}}^{e^{trial}}\right). \tag{32}$$

Thus, substituting Eq.(32) and Eq.(30) into Eq.(2) yields

$$\eta_{n+1} = \hat{\eta}\exp\left(-E_{v_{n+1}}^{e^{trial}}\right) \tag{33}$$

where $E_{v_{n+1}}^{e^{trial}} = tr\left(\mathbf{E}_{n+1}^{e^{trial}}\right)$ and $\hat{\eta} = \frac{\eta_o}{\det(\mathbf{F}_n^p)}$. In addition the linearization of Eq.(33) yields

$$d\eta_{n+1} + \hat{\eta}\exp\left(-E_{v_{n+1}}^{e^{trial}}\right) dE_v^{e^{trial}} = 0. \tag{34}$$

Now, for the correct determination of the elastoplastic tangent operator

$$\mathbb{D}^{ep} = \frac{d\bar{\tau}_{n+1}}{d\mathbf{E}_{n+1}^{e^{trial}}}, \tag{35}$$

one must add Eq.(33) to the system of equations shown in Eq.(29) and perform the linearization of this augmented system of equations. Such linearization leads to following set of equations

$$\left\{\begin{array}{l} d\mathbf{E}_{n+1}^{e} - d\mathbf{E}_{n+1}^{e^{trial}} + d\left(\Delta\lambda\right)\mathbf{N}_{\bar{\tau}_{n+1}} + \Delta\lambda \mathbf{N}_{\bar{\tau}_{n+1}} \\ d\alpha_{k_{n+1}} - d\left(\Delta\lambda\right)\mathbf{N}_{\beta_k}|_{n+1} - \Delta\lambda\, d\mathbf{N}_{\beta_k}|_{n+1} \\ df\left(\bar{\tau}_{n+1}, \beta_{k_{n+1}}, \eta_{n+1}\right) \\ d\eta_{n+1} + \hat{\eta}\exp\left(-E_{v_{n+1}}^{e^{trial}}\right) dE_v^{e^{trial}} \end{array}\right. = \left[\begin{array}{c} \mathbf{0} \\ \mathbf{0} \\ 0 \\ 0 \end{array}\right] \tag{36}$$

where

$$d\mathbf{N}_{\bar{\tau}_{n+1}} = \frac{\partial \mathbf{N}_{\bar{\tau}_{n+1}}}{\partial \bar{\tau}_{n+1}} d\bar{\tau}_{n+1} + \frac{\partial \mathbf{N}_{\bar{\tau}_{n+1}}}{\partial \alpha_{n+1}} d\alpha_{n+1} + \frac{\partial \mathbf{N}_{\bar{\tau}_{n+1}}}{\partial \eta_{n+1}} d\eta_{n+1} \tag{37a}$$

$$d\mathbf{N}_{\beta_{k_{n+1}}} = \frac{\partial \mathbf{N}_{\beta_{k_{n+1}}}}{\partial \bar{\tau}_{n+1}} d\bar{\tau}_{n+1} + \frac{\partial \mathbf{N}_{\beta_{k_{n+1}}}}{\partial \alpha_{n+1}} d\alpha_{n+1} + \frac{\partial \mathbf{N}_{\beta_{k_{n+1}}}}{\partial \eta_{n+1}} d\eta_{n+1} \tag{37b}$$

$$df\left(\bar{\tau}_{n+1}, \beta_{k_{n+1}}; \eta_{n+1}\right) = \mathbf{N}_{\bar{\tau}_{n+1}} d\bar{\tau}_{n+1} + \frac{\partial f}{\partial \alpha_{n+1}} d\alpha_{n+1} + \frac{\partial f}{\partial \eta_{n+1}} d\eta_{n+1} \tag{37c}$$

and by assuming that some elastic parameter could depend on the relative density, $\bar{\tau} = \mathbb{D}^e(\eta)\,\mathbf{E}^e$, we also impose that

$$d\mathbf{E}_{n+1}^{e} = \mathbb{D}^{e^{-1}}\left(\eta_{n+1}\right) d\bar{\tau}_{n+1} + \frac{\partial \mathbb{D}^{e^{-1}}\left(\eta_{n+1}\right)}{\partial \eta_{n+1}} \bar{\tau}_{n+1} d\eta_{n+1}. \tag{38}$$

5. MODEL CASE 1 - ELLIPTICAL OR POROUS MATERIAL MODELS

Since the work presented by Doraivelu *et. al.* (1984) many contributions have been made regarding this class of model. Some authors state that the use of the elliptical model should be used only when the relative densities are superior to 0.7, but others authors advocate that its use can be extended to lower relative densities values. Despite the discussion about the proper use of such kind of model, this model will be used here to illustrate the derivation of the consistent elastoplastic density-corrected tangent operator for elliptical models. Elliptical or porous material models are described by the following yield function functional form

$$f = AJ_2 + BI_1^2 = \sigma_\eta^2. \tag{39}$$

In this equation A and B are scalars that are, in many cases, dependent on the relative density and σ_η is the apparent yield stress. J_2 and I_1 are respectively the second invariant of the stress tensor in the deviatoric space and the first invariant of the stress tensor. In general

$$\sigma_\eta^2 = \gamma\sigma_y^2 \tag{40}$$

where σ_y is the initial yield stress of the fully dense material. Doraivelu *et al.* (1984) shown that the values of A and B are not arbitrary. However, there are a great variety of proposals for A and B in the literature. Zhdanovich (1971) proposes that a Poisson dependence on the relative density, such that

$$\nu = \frac{1}{2}\eta^n. \tag{41}$$

The exponent $n \simeq 2$ has been used to describe such dependence. The γ multiplier is known as the geometric hardening and can be also dependent on the relative density. When $\gamma = 1$ the material must behaves as a fully dense material and for some value between 0 and 1 the material should presents no mechanical strength. This value, represented by η_C, can vary for each author but is about the called tap density.

5.1 Proposal model

Let us propose that the porous material model could experience an isotropic hardening k in its dense matrix. So, the proposal material model can be represented by an yield function as

$$f = AJ_2 + BI_1^2 = \gamma\left(k + \sigma_y\right)^2. \tag{42}$$

Taking the square root in both sides of the Eq.(42) this function can be rewritten as

$$\bar{f} = S_{eq} - \gamma^{\frac{1}{2}}\left(k + \sigma_y\right) \tag{43}$$

where

$$S_{eq} = \sqrt{AJ_2 + BI_1^2} \tag{44}$$

plays the role of equivalent stress. Let us assume that isotropic hardening k of the dense material matrix can be represented by

$$k\left(\epsilon\right) = H\epsilon + \left(\sigma_\infty - \sigma_y\right)\left(1 - e^{-\delta\epsilon}\right) \tag{45}$$

where H, σ_∞ and δ are material parameters that not depend on the relative density and ϵ plays the role of the isotropic hardening strain.

5.2 Tangent operator

Once the model has been described in the previously section, it is possible now to identify the elastoplastic tangent operator for the proposal model. This identification comes from the linearization presented in Eq.(35). After a straightforward algebra manipulation it is possible to show that

$$\mathbb{D} = \mathbb{D}^{ep} = \frac{d\overline{\tau}_{n+1}}{d\mathbf{E}_{n+1}^{etrial}} = \mathbb{D}_{std}^{ep}\eta_{cor} \tag{46}$$

where

$$\mathbb{D}_{std}^{ep} = \left[\mathbb{D}^e\left(\eta_{n+1}\right)^{-1} + \Delta\lambda\frac{\partial\mathbf{N}_{\overline{\tau}_{n+1}}}{\partial\overline{\tau}_{n+1}} - \frac{1}{\frac{\partial\bar{f}}{\partial\epsilon_{n+1}}\sqrt{\gamma_{n+1}}}\left(\mathbf{N}_{\overline{\tau}_{n+1}} \otimes \mathbf{N}_{\overline{\tau}_{n+1}}\right)\right]^{-1} \tag{47}$$

$$\eta_{cor} = \mathbb{I} - \eta_{n+1}\left[\mathbb{D}^e\left(\eta_{n+1}\right)^{-1}\frac{\partial\mathbb{D}^e(\eta_{n+1})}{\partial\eta_{n+1}}\mathbf{E}_{n+1}^e + f_c\mathbf{N}_{\overline{\tau}_{n+1}} - \Delta\lambda\frac{\partial\mathbf{N}_{\overline{\tau}_{n+1}}}{\partial\eta_{n+1}}\right] \otimes \mathbf{I} \tag{48}$$

with

$$f_c = \frac{\frac{\partial \bar{f}}{\partial \eta_{n+1}}}{\frac{\partial \bar{f}}{\partial \epsilon_{n+1}} \sqrt{\gamma_{n+1}}} + \frac{\Delta\lambda}{2\gamma_{n+1}} \frac{\partial \gamma(\eta_{n+1})}{\partial \eta_{n+1}}. \tag{49}$$

6. MODEL CASE 2 - SMOOTH THREE SURFACE *Cap* MODEL

This *Cap* model, originally proposed by Swan & Seo (2000), is modified in order to be consistent with the proposed finite deformation description and to account for the dependency on the material parameters to the relative density of the material. The *Cap* description presented here is very concise. More details can be found in Rossi & Alves (2008).

The smooth *Cap* model is composed by a exponential shear failure surface f_1, a compression circular *Cap* f_2 and a tension circular *Cap* f_3. These yield functions are given by

$$f_1 \left(\bar{\tau}, \bar{\chi}^D\right) = \left\|\bar{\mathbf{S}}^D\right\| - F_e \left(I_1\right) \le 0 \tag{50}$$

$$f_2 \left(\bar{\tau}, \bar{\chi}^D, \omega\right) = \left\|\bar{\mathbf{S}}^D\right\|^2 - F_c \left(I_1, \omega\right) \le 0 \tag{51}$$

$$f_3 \left(\bar{\tau}, \bar{\chi}^D\right) = \left\|\bar{\mathbf{S}}^D\right\|^2 - F_t \left(I_1\right) \le 0. \tag{52}$$

where ω is the center of the compression *Cap*, $I_1 = tr\left(\bar{\tau}\right)$, and the tensor $\bar{\mathbf{S}}^D$ is given by

$$\bar{\mathbf{S}}^D = \bar{\tau}^D - \bar{\chi}^D \tag{53}$$

where $\left\|\bar{\mathbf{S}}^D\right\| = \sqrt{\bar{\mathbf{S}}^D \cdot \bar{\mathbf{S}}^D}$, $\bar{\chi}^D$ denotes the back stress and $\left(\circ\right)^D$ represents the deviatoric part of $\left(\circ\right)$. At this point, one assumes F_e, F_c and F_t to be

$$F_e \left(I_1\right) = \alpha + \gamma \left[1 - e^{\beta I_1}\right] \quad if \quad I_1^c \le I_1 \le I_1^T \tag{54}$$

$$F_c \left(I_1, \omega\right) = R^2 \left(\omega\right) - \left(I_1 - \omega\right)^2 \; if \quad I_1 \le I_1^c \tag{55}$$

$$F_t \left(I_1\right) = R_T^2 - I_1^2 \qquad if \quad I_1 \ge I_1^T. \tag{56}$$

in which α, β and γ are material parameters, I_1^c denotes the intersection point between the tension *Cap* and the Drucker-Prager exponential envelop and I_1^T the intersection point between the compression *Cap* and the Drucker-Prager exponential envelop. Here, R and R_T are the radius of the compression and tension *Caps* respectively. In this work we assume that the powder can be modeled by an associative flow rule an that the normal dissipation hypothesis is valid. Thus, inside of the framework of the existence of a local intermediate configuration, we introduce the modified plastic stretching $\bar{\mathbf{D}}^p$, which measures the rate of the plastic deformation, as

$$\bar{\mathbf{D}}^p = \sum_j \dot{\lambda}_j \frac{\partial f_j}{\partial \bar{\tau}} \tag{57}$$

and the *Karush-Kuhn-Tucker* conditions given by

$$f_j \le 0, \quad \dot{\lambda}_j \ge 0 \quad \text{and} \quad f_j \dot{\lambda}_j = 0. \tag{58}$$

In this context, the stress space is defined by $\mathcal{E} = \left\{\bar{\tau} \middle| f_i\left(\bar{\tau}, \beta, \eta\right) \le 0\right\}$. The loading conditions associated with the *Cap* model are identified as: Hyperelastic loading ($f_j < 0$, $j = 1...3$); Loading occurring through the shear surface ($f_1 = 0$ and $\dot{\lambda}_1 > 0$); Loading occurring through the compression *Cap* surface ($f_2 = 0$ and $\dot{\lambda}_2 > 0$); Loading occurring through the tension *Cap* surface ($f_3 = 0$ and $\dot{\lambda}_3 > 0$); Surfaces 1 and 2 are active simultaneously (i.e., $f_1 = f_2 = 0$ and $\dot{\lambda}_1 > 0$ and $\dot{\lambda}_2 > 0$); Surfaces 1 and 3 are active simultaneously (i.e., $f_1 = f_3 = 0$ and $\dot{\lambda}_1 > 0$ and $\dot{\lambda}_3 > 0$).

The evolution of the center of the compression *Cap* is given by

$$\dot{\omega} = h'\left(\omega\right) tr\left(\bar{\mathbf{D}}^p\right) \tag{59}$$

where h' is the tangent hardening modulus expressed as

$$h'\left(\omega\right) = \frac{dh\left(\omega\right)}{d\omega} = \frac{e^{-D\xi(\omega)}}{WD\xi'\left(\omega\right)} \tag{60}$$

with D and W being material parameters and

$$\xi\left(\omega\right) = \omega - R\left(\omega\right). \tag{61}$$

The kinematics hardening

$$\dot{\bar{\chi}}^D = C\mathbb{I}^D \bar{\mathbf{D}}^p, \tag{62}$$

where C is the constant hardening parameter and $\mathbb{I}^D = \mathbb{I} - \frac{1}{3}\left[\mathbf{I} \otimes \mathbf{I}\right]$.

6.1 Consistent density-corrected tangent operators

1. Drucker-Prager exponential envelope f_1 is active: The tangent operator relating the Drucker-Prager exponential envelope, disregarding the kinematic hardening, can be derived by the linearization of the return mapping equations related with this surface. It yields:

$$\mathbb{D}_1 = \mathbb{D}_1^{ep} = \frac{d\overline{\tau}_{n+1}}{d\mathbf{E}_{n+1}^{e^{trial}}} = \mathbb{D}_{1_{std}}^{ep} \eta_{1_{cor}} \tag{63}$$

with

$$\mathbb{D}_{1_{std}}^{ep} = \left[\mathbb{D}^{e^{-1}} + \Delta\lambda_1 \left(\left\| \overline{\tau}_{n+1}^D \right\|^{-1} \mathbb{I}^D - \mathbf{N}_{n+1} \otimes \mathbf{N}_{n+1} \right) + \gamma\beta e^{\beta I_{1_{n+1}}} (\mathbf{I} \otimes \mathbf{I}) \right]^{-1} \tag{64}$$

$$\eta_{1_{cor}} = \mathbb{I} - \eta_{n+1} \mathbb{D}^{e^{-1}} \frac{\partial \mathbb{D}^e}{\partial \eta_{n+1}} \mathbf{E}_{n+1}^e \otimes \mathbf{I}. \tag{65}$$

2. Compression *Cap* f_2 is active: In the powder compaction processes, the compression *Cap* plays a very important role. In such forming process the material is confined inside a mold and then submitted to high compression loadings. Therefore, the negative hydrostatic stress portion tends to be more prominent. Two are the requirements imposed on this surface: the *Cap* must be centered in $\left(I_1 = \omega, \left\| \bar{\mathbf{S}}^D \right\| = 0\right)$ and that a smooth intersection with the Drucker-Prager exponential envelope exists. The tangent operator, disregarding the kinematics hardening, yields

$$\mathbb{D}_2^{ep} = \frac{d\overline{\tau}_{n+1}}{d\mathbf{E}_{n+1}^{e^{trial}}} = \mathbb{D}_{2_{std}}^{ep} \eta_{2_{cor}} \tag{66}$$

with

$$\mathbb{D}_{2_{std}}^{ep} = \left[\mathbb{D}^{e^{-1}} + 2\Delta\lambda_2 \left[\mathbb{I}^D + \mathbf{I} \otimes \mathbf{I}\right] - C_1 \frac{\partial f_2}{\partial \overline{\tau}_{n+1}} \otimes \mathbf{I} - \frac{1}{\frac{\partial f_2}{\partial \omega_{n+1}}} \left(C_2 \frac{\partial f_2}{\partial \overline{\tau}_{n+1}} - 2\Delta\lambda_2 \mathbf{I} \right) \otimes \frac{\partial f_2}{\partial \overline{\tau}_{n+1}} \right]^{-1} \tag{67}$$

$$\eta_{2_{cor}} = -\eta_{n+1} \left[\mathbb{D}^{e^{-1}} \frac{\partial \mathbb{D}^e}{\partial \eta_{n+1}} \mathbf{E}_{n+1}^e \otimes \mathbf{I} + \frac{1}{\frac{\partial f_2}{\partial \omega_{n+1}}} \frac{\partial f_2}{\partial \eta_{n+1}} \left(C_2 \frac{\partial f_2}{\partial \overline{\tau}_{n+1}} - 2\Delta\lambda_2 \mathbf{I} \right) \otimes \mathbf{I} \right] + \mathbb{I} \tag{68}$$

where

$$C_1 = \frac{\Delta\lambda_2}{I_{1_{n+1}} - \omega_{n+1}} \qquad C_2 = \frac{1 + 6\Delta\lambda_2 h'_{n+1}}{6h'_{n+1}\left(I_{1_{n+1}} - \omega_{n+1}\right)} \tag{69}$$

3. Tension fixed *Cap* f_3 is active: Similarly to the compression *Cap*, there are requirements imposed on this surface, i.e, the tension *Cap* must be centered in $\left(I_1 = 0, \left\| \bar{\mathbf{S}}^D \right\| = 0\right)$, a smooth intersection with the Drucker-Prager exponential envelope must take place and that this *Cap* remains fixed, that is, there are no isotropic hardening parameter associated with this *Cap*. Again, disregarding the kinematics hardening and taking the linearization of Eq.(**??**) together with Eq.(33) yields

$$\mathbb{D}_3^{ep} = \frac{d\overline{\tau}_{n+1}}{d\mathbf{E}_{n+1}^{e^{trial}}} = \mathbb{D}_{3_{std}}^{ep} \eta_{3_{cor}} \tag{70}$$

with

$$\mathbb{D} = \left\{ \mathbb{D}^{e^{-1}} + 2\Delta\lambda_3 \left[\mathbb{I}^D + \mathbf{I} \otimes \mathbf{I}\right] \right\}^{-1} \qquad \eta_{3_{cor}} = \mathbb{I} - \eta_{n+1} \mathbb{D}^{e^{-1}} \frac{\partial \mathbb{D}^e}{\partial \eta_{n+1}} \mathbf{E}_{n+1}^e \otimes \mathbf{I}. \tag{71}$$

7. RESULTS

In order to attest the effect of the correction η_{cor} into the standard elastoplastic tangent operator $\mathbb{D}_{std}^{ep}$ it is proposed in this section the simulation of a unitary sided body under axisymmetric assumption, Figure 1*a),* representing an isostatic compaction and the simulation of the model under plane strain assumption, Figure 1*b),* representing an uniaxial compaction. The discretization of the body as well as their boundary conditions are also displayed in Figure 1.

- Model 1 - Elliptical model: In this example we consider a material model as described by Eq.(43) and Eq.(45) with A, B and γ given by the Doraivelu *et al.* (1984) model. The simulation consists in: Set the initial relative density is assumed to be $\eta_o = 0.7$; Set the material parameters to be: $\eta_C = 0.4$, $E = 10000MPa$, $\nu = 0.1$, $H = 130MPa$, $\delta = 17$, $\sigma_\infty = 715MPa$ and $\sigma_y = 100MPa$; Carry out the isostatic compaction by applying simultaneously $u_r = u_z = -0.112096$ in 20 equally spaced steps; Carry out the uniaxial compaction by applying $u_y = -0.3$ in 20 equally spaced steps.

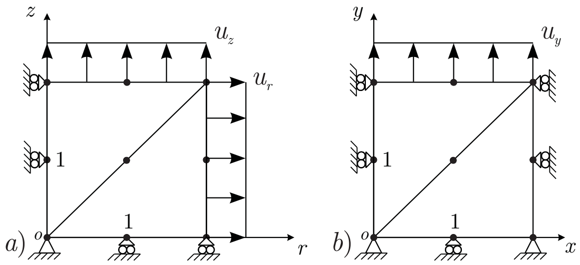

Figure 1. Model problem - *a)* Isostatic compaction - Axisymmetric model - *b)* Uniaxial compaction - Plane strain model.

- Model 2 - Smooth *Cap* Model: Set initial relative density set to $\eta_o = 0.4$; Set the material parameters to be: $E(\eta) = 3640\eta^{3.9}$, see Gethin *et al.* (1995), $\nu = 0.35$, $W = 1$, $D = 7 \times 10^{-3} MPa^{-1}$,$\omega_o = -0.2MPa$, $\alpha = 5.883MPa$, $\beta = 10^{-3}$ and $\lambda = 226.455$; Carry out the isostatic compaction by applying simultaneously $u_r = u_z = -0.263194$ in 100 equally spaced steps; Carry out the uniaxial compaction by applying $u_y = -0.6$ in 100 equally spaced steps.

The Figure 2 shows the comparison between the convergence results expressed in terms of the residue norm $\|\mathbf{r}_{n+1}\|_\infty$ versus the number of iterations n_{iter} in a one axis log graph. The results presented in this figure take into account the use of the standard consistent elastoplastic tangent operator $\mathbb{D}^{ep}_{std}$ and density-corrected $\mathbb{D}^{ep} = [\mathbb{D}^{ep}_{std}]^{-1}\eta_{cor}$ under the same load step. More specifically, figure 2a) and b) shows the convergence results for the model 1 under isostatic and uniaxial compression respectively and figure 2c) and d) shows the convergence results for the model 2 under the same boundary conditions.

The residue vector is the classical one used in the finite element analysis context, $\mathbf{r} = \mathbf{f}^{ext} - \mathbf{f}^{int}$. In both cases the criterion of convergence is the number of iterations to achieve the admissible error given by $\|\mathbf{r}_{n+1}\|^{adm}_\infty \leq 10^{-6}$. In all cases it is employed a *Newton-Raphson* method with line search.

8. CONCLUSION

This paper deals with the analytical derivation of the consistent density-corrected tangent operators for density-dependent finite plasticity models in the framework of the *Total Lagrangian* formulation, multiplicative finite strain plasticity, logarithmic strains and the exponential return mapping algorithms. It is clearly shown that the density dependence of the material model implies in corrections on the standard elastoplastic tangent operators. Based on the results shown it is possible note that if no density correction is performed on standard elastoplastic tangent operators the convergence is affected, increasing dramatically the number the iterations to reach the specified admissible error, or even leading to non convergence state. On the other hand, if the correction is taken into account then the convergence is slightly affected during the simulation. The way the convergence is affected depends on the nonlinear constitutive model density evolution dependence.

9. ACKNOWLEDGMENTS

The authors wish to acknowledge the support of the CNPq, Conselho Nacional de Desenvolvimento Científico e Tecnológico of Brazil. Grant number 473343/2008-8.

References

Doraivelu, A.M., Gegel, H.L., Gunasekera, J.S., Malas, J.C., Morgan, J.T., 1984, "A new yield function for compressible P/M materials", International Journal of Mechanical Sciences, Vol. 26, No.9-10, pp. 527-535.

Eterovic, A.L., Bathe, K.J., 1990, "A Hyperelastic-Based Large Strain Elasto-Plastic Constitutive Formulation with Combined Isotropic-Kinematic Hardening using the Logarithmic Stress and Strain Measures", International Journal for Numerical Methods in Engineering, Vol.30, issue 6, pp. 1099-1114.

Gethin, DT, Lewis, RW, Ariffin, AK. Modelling compaction and ejection processes in the generation of green powder compacts. In ASME International Mechanical Engineering Congress and Exposition, San Francisco, ADM-216, 1995; 27–45.

Ortiz, M., Martin, J.B., 1989, "Symmetry-preserving return mapping algorithms and incrementally extremal paths: a unification of concepts", International Journal for Numerical Methods in Engineering, Vol. 28, pp. 1839–1853.

Ortiz, M., Radovitzky, R.A., Repeto, E.A., 2001, "The computation of exponential and logarithmic mappings and their fisrt and second linearizations", International Journal for Numerical Methods in Engineering, Vol.52, pp. 1431-1441.

Pérez-Foguet, A., Rodríguez-Ferran, A., Huerta, A., 2000, "Numerical differentiation for local and global tangent operators in computational plasticity", Computer Methods in Applied Mechanics and Engineering, Vol. 189, No. 1, pp. 277-296.

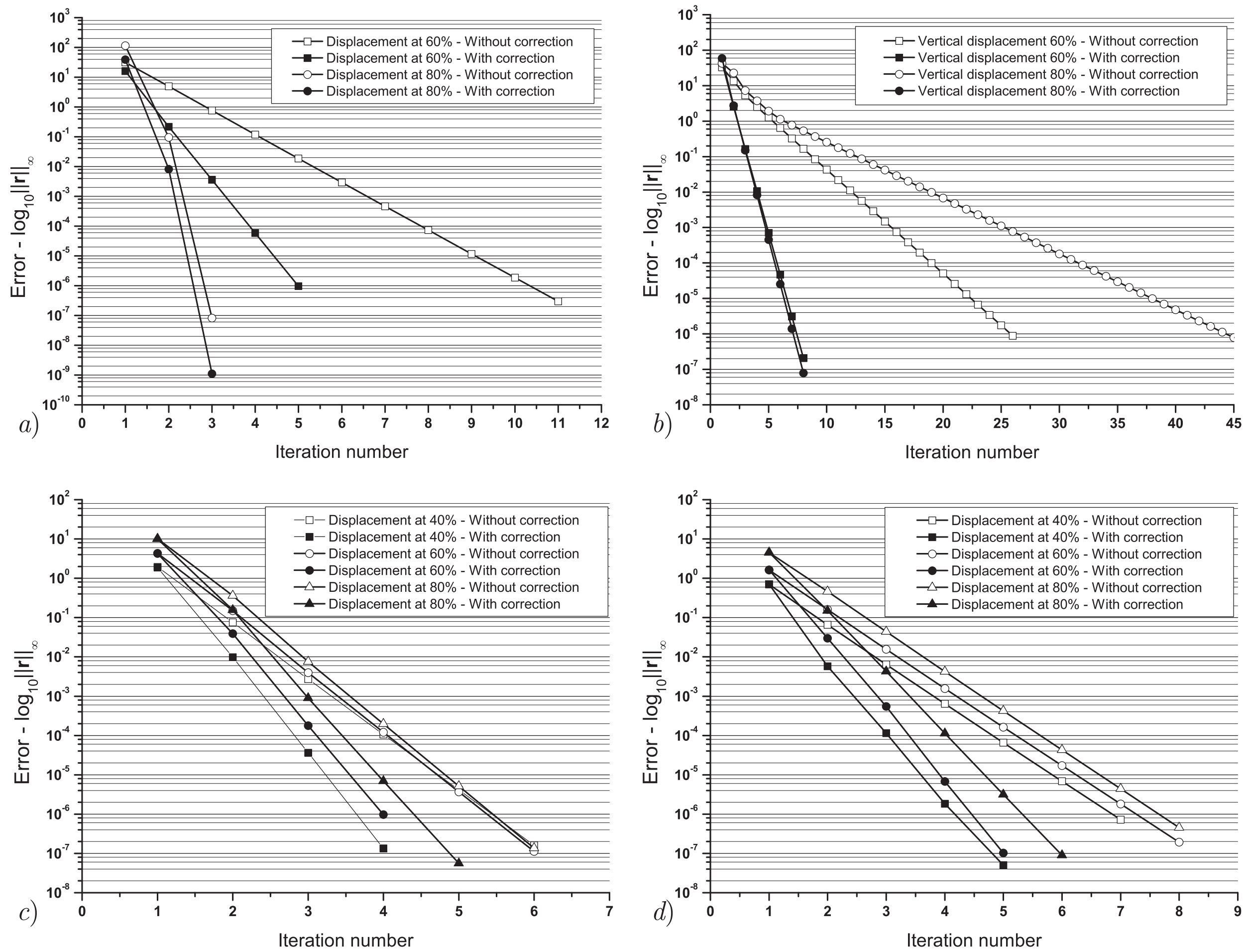

Figure 2. Convergence comparison analysis: *a)* Model 1 - Isostatic compaction - *b)* Model 1 - Uniaxial compaction - *c)* Model 2 - Isostatic compaction - *d)* Model 2 - Uniaxial compaction.

Pérez-Foguet, A., Rodríguez-Ferran, A., Huerta, A., 2001, "Consistent tangent matrices for density-dependent finite plasticity models", International Journal for Numerical and Analytical Methods in Geomechanics, Vol. 25, No. 11, pp. 1045-1075.

Pérez-Foguet, A., Rodríguez-Ferran, A., Huerta, A., 2003, "Efficient and accurate approach for powder compaction problems", Computational Mechanics, Vol. 30, No. 3, pp. 220-234.

Rossi, R., Alves, M. K., Al-Qureshi, H. A., 2008, “A total Lagrangian framework for simulation of powder compaction process based on a smooth three-surface Cap model and a mesh-free method”, International Journal for Numerical Methods in Engineering, Vol. 75, No.12, pp. 1457-1491, 2008.

Simo, J.C., 1992, "Algorithms for static and dynamic multiplicative plasticity that preserve the classical return mapping schemes of infinitesimal theory", Computer Methods in Applied Mechanics and Engineering, Vol. 99, pp. 61-112.

Souza Neto, E.A., Peric, D., Owen,D.R.J., 1998, "Continuum modeling and numerical simulation of material damage at finite strains", Archives of Computational Methods in Engineering, Vol. 5, pp. 311-384.

Swan, C.C., Seo, Y.K., "A smooth three-surface elasto-plastic cap model: rate formulation, integration algorithm and consistent tangent operator", To be published.

Swan, C.C, Seo, Y.K., 2000, "A smooth cap plasticity model for soils: Integration algorithm, parameter estimation and applications", Tech. Rep. No. 2000-043, Ctr. for Comp. Aided Des., University of Iowa, Iowa.

Weber, G., Anand, L., 1990, "Finite deformation constitutive equations and a time integration procedure for isotropic hyperelastic-viscoplastic solids", Computer Methods in Applied Mechanics and Engineering, Vol.79, issue 2, pp. 73-202.

Zhdanovich, G.M., 1971, "Theory of compaction of metal powder", translated from Teorize Pressovaniya Metzllichaskikli Poroshkov,1969, p. 1-262, by the Foreign Technology Division, Wright-Patterson Air Force Base, OH.

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