Numerical solution of a two-equation turbulence model for bubbly boundary layers at low void fractions

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Abstract. *This work discusses the lower boundary condition for bubbly boundary layers at low void fractions. Local solutions developed for the fully turbulent region for the mean velocity, turbulent kinetic energy and dissipation rate are implemented in the finite volume numerical code TURBO2D. Effects of the local flow properties including the void fraction, wall shear stress, slip velocity and two empirical parameters are accounted for by the proposed model. The work shows also how one of the standard single-phase constants in the* ϵ*-equation,* $C_{\epsilon 1}$*, needs to be modified to include the effects of the bubbles. The theory is compared with the experimental data of Marié et al. (IJMF, 23, 227-247, 1997) and with the previous formulation of Troshko and Hassan (IJHMT, 44, 871-875, 2001a).*

Keywords: *bubbly flow, law of the wall, kappa-epsilon model.*

1. INTRODUCTION

The advances toward a realistic three-dimensional representation of multiphase flows have been significant over the last twenty years. As computer power increases and experimental methods become more sophisticate, appropriate constitutive relations can be developed and implemented in predictive codes.

On this note, different approaches to model two-phase disperse flows can be routinely found in literature. One possible method is to describe the time-dependent three-dimensional motion of bubbles individually whereas the liquid phase is described by the averaged Navier-Stokes equations. Alternatively, the concept of interpenetrating continua can be used and both phases are treated in terms of two sets of conservation equations. However, irrespective of the method chosen, the separate treatment of the phases means that their mutual interaction needs to be also modelled.

The use of discrete bubble models is typified in the works of Delnoij et al. (1997), Xu et al. (2002) and Lu et al. (2005) where flows in gas-liquid bubble columns and in boundary layers are studied. Two-fluid models have been preferred by Lee et al. (1989) and Lopez de Bertodano et al. (1994). The work of Lu et al. (2005) is particularly interesting since the effects of few large bubbles injected near the wall are studied by direct numerical simulations (DNS). Thus, the flow details are fully resolved including the shape of bubbles and the flow around them. A major difficulty with DNS simulations, however, is the small Reynolds number and domain size that can be achieved.

Two-fluid models on the other hand can be easily applied to industrial problems since the macroscopic effects of the interaction between phases are represented through constitutive equations. Several notable examples are found in the literature. Drew and Lahey (1982) applied an algebraic model to study phase distribution in vertical bubbly pipe flow. To account for non-isotropic effects, Lopez de Bertodano (1990) used a τ-ϵ model. This model specifies transport equations to all components of the Reynolds stress tensor for the liquid phase. Troshko and Hassan (2001b) proposed a new model for the liquid phase and a new logarithmic law of the wall.

The treatment of the wall boundary condition for bubbly flows is an important issue that has been poorly discussed in all previous works. Because of the large gradients that are found near a wall, is it normally impractical to specify sufficiently fine computational grids capable of capturing this behaviour. The common practice, then, is to specify the lower boundary condition in the fully turbulent region through a local solution, a logarithmic mean velocity solution.

Early works on two-phase flows have assumed the single-phase boundary condition to remain valid (Lopez de Bertodano, 1990). However, some posterior experimental evidence has shown otherwise. Moursali et al. (1995) and Marié et al. (1997) studied flows developing over a vertical, smooth, flat plate in the presence of small bubbles. In Moursali et al. (1995) data on void fraction distribution, wall shear stress and liquid mean velocity profiles were presented for different mean bubble diameter. Depending on their sizes, a migration phenomenon was observed with a resulting accumulation and deceleration of bubbles in the near wall region. This process resulted in an increase of skin-friction and the modification of the classical law of the wall. Marié et al. (1997) used simple analytical considerations and dimensional analysis to propose a modified law of the wall. They also presented longitudinal turbulence intensity profiles and showed that turbulence is increased by two main mechanisms: a modification of the wall production and the creation of pseudo-turbulence in the external layer.

Some of the previous efforts aimed at developing near wall solutions for bubbly flows were reviewed by Troshko and Hassan (2001a), who in turn proposed a new expression for the logarithmic law. Their analysis includes expressions for the mean velocity profile, the turbulent kinetic energy and the dissipation rate. The turbulent viscosity concept was used to find the local solution. The non-linear interaction between shear and bubble induced turbulent fields was accounted for by a proportionality coefficient.

Using the intermediate variable technique, Bitencourt et al. (2008) derived alternative boundary conditions to bubbly flows. Their flow local solutions were not obtained through dimensional arguments but through a direct integration of the equations of motion. New solutions were proposed for the mean velocity profile, turbulent kinetic energy and dissipation rate. Results were tested against the experiments of Marié et al (1997).

The purpose of the present work is to implement the lower boundary conditions of Bitencourt et al. (2008) in the computational fluid dynamics program TURBO2D. Model predictions are then compared with other bubbly flow models and experiments.

2. THEORY

The time-averaged three-dimensional two-fluid conservation equations have been introduced by Ishii (1975). With no interfacial mass source, the mass conservation equations can be written as

$$\frac{D_k \alpha_k}{Dt} + \alpha_k \nabla \cdot \overline{u_k} = 0 \tag{1}$$

where k refers to the phase under consideration and the notation is classical.

The momentum equations can be written as

$$\alpha_k \rho_k \frac{D\overline{u_k}}{Dt} = \alpha_k \left(\nabla \cdot \overline{T_k} + \rho_k g \right) - \left(\overline{T_{ki}} - \overline{T_k} \right) \cdot \nabla \alpha_k + M_{ki} \tag{2}$$

where, neglecting the viscous stresses, the stress tensor for phase k is given by

$$\overline{T_k} = -p_k I - \rho_k \overline{u'_k u'_k}, \tag{3}$$

p_k is the static pressure, g is the gravitational acceleration, M_{ki} is the interfacial force on phase k and $-\rho_k \overline{u'_k u'_k}$ is the Reynolds stress tensor.

Equation (2) can only be solved provided the interfacial forces and the Reynolds stress components are modelled and appropriate boundary conditions are furnished.

Concerning the interfacial forces, the simplest possible approach is to consider the gas bubbles as mere voidages, so that no transfer of momentum occurs in the gas phase and, therefore, the flow dynamics is entirely determined by the liquid phase. Turbulence in the liquid phase is decomposed into contributions due to shear and to bubble agitation. This latter assumption is considered valid for void fraction levels below 10% (Lance and Bataille, 1991).

The Reynolds stress tensor is modelled through the eddy viscosity concept, so that

$$-\rho_k \overline{u'_k u'_k} = \rho_k \left(\nu_k^t \left(\nabla \overline{u_k} + (\nabla \overline{u_k})^T - \frac{2}{3} I \left(\nabla \cdot \overline{u_k} \right) \right) - \frac{2}{3} I \kappa_k \right) \tag{4}$$

and κ is the turbulent kinetic energy.

Considering that shear and bubble induced turbulence effects can be superimposed, it follows immediately that

$$\nu_k^t = \nu_{ks}^t + \nu_{kb}^t, \tag{5}$$

with ν_{ks}^t = eddy viscosity due to shear; ν_{kb}^t = eddy viscosity due to bubble agitation.

Different approaches can be used to specify the eddy viscosities ν_{ks}^t and ν_{kb}^t. The shear induced viscosity can be modelled through the mixing-length hypothesis, that is,

$$\nu_{ks}^t = l_m^2 \left| \nabla \overline{u_k} \right|, \tag{6}$$

where $l_m (= \varkappa y)$ is the mixing length and $\varkappa$ (= 0.4) is the von Kármán's constant.

An alternative is to consider, the κ-ϵ model so that

$$\nu^t_{ks} = c_\nu \frac{\kappa^2_k}{\epsilon_k}, \tag{7}$$

and the turbulent kinetic energy, κ_k, and the dissipation rate of turbulent kinetic energy, ϵ_k, are given by transport equations.

In the following, the magnitude of the bubbles Reynolds stress will be considered much smaller than the liquid phase so the the phasic subscript k will be dropped from all equations.

Thus, the standard equations for κ and ϵ can be written as

$$\frac{D\kappa}{Dt} = P - \epsilon + \frac{\partial}{\partial y}\Big(\frac{\nu^t}{\sigma_\kappa}\frac{\partial \kappa}{\partial y}\Big), \tag{8}$$

$$\frac{D\epsilon}{Dt} = c_{\epsilon_1}\frac{\epsilon}{\kappa}P - c_{\epsilon_2}\frac{\epsilon^2}{\kappa} + \frac{\partial}{\partial y}\left(\frac{\nu^t}{\sigma_\epsilon}\frac{\partial \epsilon}{\partial y}\right), \tag{9}$$

$$P = -\langle uv \rangle \left(\frac{\partial U_l}{\partial y}\right), \tag{10}$$

where all the c's and σ's are model constants. Typical values of the empirical constants for single phase flows are shown in Table 1.

Table 1. Model constants for single phase flows.

c_ν	c_{ϵ_1}	c_{ϵ_2}	σ_κ	σ_ϵ
0.09	1.44	1.92	1.0	1.30

The bubble induced turbulence is suggested by Sato et al. (1981) to be modelled by accounting for the drift phenomena of liquid particles due to the relative motion of gas bubbles. The result is

$$\nu^t_b = \varkappa_l \alpha_{g_{max}} y u_R, \tag{11}$$

where $\varkappa_l$ is an empirical constant (= 1.2 to 1.4), $\alpha_{g_{max}}$ the peak of gas void fraction and u_R the mean relative velocity of the bubbles.

The system of equations (2) to (11) needs to be complemented by appropriate boundary conditions. This is normally achieved with the specification of wall functions. We shall see this next.

Bitencourt et al. (2008) have used the intermediate variable technique to find a local solution for the near wall region of bubbly flows. The effects of the near wall changes in void fraction are represented by parameter β. The resulting expressions are

$$\overline{u^+} = \frac{\beta}{\varkappa}\ln(y^+) + B^+, \tag{12}$$

where the wall variables are defined as $\overline{u^+} = \overline{u^+}/u_*$ and $y^+ = yu_*/\nu$, and

$$\beta = \frac{\varkappa_l \alpha_{g_{max}} U_R}{2\varkappa u_*}\Big(\sqrt{1 + \frac{(2\varkappa u_*)^2}{(\varkappa_l \alpha_{g_{max}} U_R)^2(1-\alpha_{g_{max}})}} - 1\Big), \tag{13}$$

$$\kappa = \frac{\beta u_*^2}{\sqrt{c_\nu}}, \tag{14}$$

$$\epsilon = \frac{\beta u_*^3}{\varkappa y}, \tag{15}$$

In additon, Bitencourt et al. (2008) propose that the c_{ϵ_1} parameter in Eq. (9) be modified according to

$$c_{\epsilon_1} = c_{\epsilon_2} - \frac{1}{\sigma_\epsilon} \frac{\kappa^2}{\sqrt{c_\nu}} \frac{1}{\beta}. \tag{16}$$

This is an important result that has been shown in Bitencourt et al. (2008) for the first time. In the limit as β tends to unit, Eq. (16) reduces to the single-flow equation for c_{ϵ_1}.

Previous studies have show that free turbulent shear flows are very sensitive to changes in c_{ϵ_1} and c_{ϵ_2}. Variations of 10% in their values might result in changes of about 40% in the growth rate of a shear layer. In general, σ_ϵ is fixed so that adequate adjustments on c_{ϵ_1} are made to the flows of interest.

In a two-phase flow, Eq. (16) shows that c_{ϵ_1} must be corrected to account for the action of the bubbles. Furthermore, Eq. (16) shows that this correction must be made in terms of β^{-1}.

3. COMPUTATIONAL DETAILS

The numerical simulations were performed with the code Turbo-2D (Fontoura Rodrigues (1990)), which is a two-dimensional code based on the finite elements method. The application of standard Garlekin discretization to problems that are dominated by convection, frequently leads to non-physical oscillations and convergence difficulties. To alleviate this tendency, code Turbo-2D resorts to the balance dissipation method proposed by Hughes and Brooks (1979) and Kelly et al. (1980) and implemented by Brun (1988). The structure of code Turbo-2D was based on the work of Brison et al. (1985), which uses finite elements of type P1-isoP2 for space discretization and semi-implicit time discretization.

The governing equations are discretized in space through triangular finite elements, defined by linear interpolation functions. The compatibility conditions between pressure and velocity fields are preserved by using two calculation meshes. The pressure field is calculated with a mesh with P1-type elements. Velocity and all other variables are calculated using a P1-isoP2 mesh, constructed from the P1 mesh by dividing one segment into two. This procedure generates four P1-isoP2 elements from one P1 element.

Temporal discretization of the governing equations is made through a sequential semi-implicit finite difference algorithm. This algorithm uses an error of truncation which permits the linearization of the system of equations at each time step through to the following procedure: (i) at instant $t = n\Delta t$ the mean velocity, pressure, turbulent kinetic energy and dissipation rate fields are known, (ii) at $t = (n+1)\Delta t$ the velocity and pressure fields are calculated by solving the coupled equations of continuity and momentum. Sequentially, the turbulent kinetic energy and the dissipation rate are calculated for the same instant, i.e. $t = (n+1)\Delta t$.

A successful simulation of the flow under scrutiny depends, of course, on the correct specification of the boundary conditions. Here, the inflow values of the mean velocity and of the turbulence kinetic energy were taken directly from the experimental data. In the region adjacent to the surface, wall functions were used as explained next. At the top, symmetry condition was used. For the outflow, symmetry (zero normal gradient) conditions were applied.

Since the standard κ-ε turbulence model does not hold for low values of the turbulent Reynolds number, a common practice is to use wall functions to express the flow behaviour in the near wall region. In finite elements, the mesh does not reach the wall. Thus, the velocity tangent to the solid wall has to be specified as a function of the distance from the wall, d.

Clearly, the chosen value of d where the boundary conditions are to be applied must be selected so that d^+ (=du_*/ν) lies within the range of validity of the law of the wall. Thus, *a posteriori* computations of d^+ have to be performed. In many cases, computational decisions and meshing procedures do have an impact on the accuracy of numerical predictions. For most finite element codes, acceptable values of d^+ obey the relation $d^+ < 100$ in order to prevent numerical instabilities. For attached flows, the best results are normally found for $30 < d^+ < 50$. In the present algorithm, d is informed as an initial value. Normally, computations are started with small values of d. This value is then progressively increased until a maximum converged value is obtained. Ideally, the selected value of d should satisfy $30 < d^+$. This condition, however, normally can only be satisfied for attached flows. The ideal d is, in any case, always determined by trial-and-error.

During calculations, u and u_* at a given iteration are found through a system of non-linear equations. The explicit treatment of this non-linearity causes heavy numerical instabilities, independently of the type of law of wall that is adopted. Thus, the introduction of a stabilization scheme for the calculation of u_* by a sub-relaxation method is in order. Turbo-2D uses an iterative minimum residual algorithm to find u_* that preserves code stability. The minimization algorithm was particularly developed so as to implement law of the wall formulations that are appropriate to the description of flows subject to an adverse pressure gradient. This very sophisticated procedure will be described in detail elsewhere.

The computations were performed with a very fine mesh with 13888 nodes (P1-isoP2). Here we should point out to the reader a mesh with 13888 nodes is considered to be extremely fine for finite elements standards.

4. RESULTS

The present simulation is validated against the data of Marié et al. (1997). These authors have studied experimentally the wall region of a turbulent boundary layer developing over a vertical flat plate. In addition to mean velocity data, void fraction, wall shear stress and longitudinal turbulent intensity profiles were reported. To the best of our knowledge, the data set of Marié et al. (1997) constitutes the best account of bubbly boundary layer flow that can be found in literature.

For the sake of completeness, the theory of Troshko and Hassan (2001a) will also be tested here. These authors define β as

$$\beta_{TH} = \left[(1-\alpha_{g_{max}})\left(1+\alpha_{g_{max}}\frac{\varkappa_l}{\varkappa}\frac{U_R}{u_*}\right)\right]^{-1} \tag{17}$$

The bubbles slip velocity is evaluated from (Ishii and Zuber, 1979)

$$U_R = [4g\sigma\Delta\rho/\rho_l^2]^{1/4}(1-\alpha_{g_{max}})^{1.75}, \tag{18}$$

where σ is the surface tension and $\Delta\rho$ is the density difference of the phases.

The coefficient $\varkappa_l$ was determined by Troshko and Hassan (2001a) through a direct fitting of the experimental data to Eq. (12) with β evaluated from (17). The resulting expression was

$$\varkappa_{l_{TH}} = 4.9453\, exp(-40.661 u_*), \tag{19}$$

where the friction velocity is given in ms^{-1}.

The following tables show the physical properties and the flow conditions used in the present comparison, where U_δ stands for the external boundary layer velocity.

Table 2. Physical properties of fluids.

ρ_{water} $[kg/m^3]$	ρ_{air} $[kg/m^3]$	g $[m/s^2]$	σ $[N/m]$
1000	1.225	9.81	0.073

Table 3. Flow properties.

U_δ $[ms^{-1}]$	$\alpha_{g_{max}}$	u_* $[ms^{-1}]$
1	0.016	0.047
1	0.038	0.049
1	0.068	0.052

Predictions for the mean velocity profile are shown in Figs. 1 to 3. Seven curves can be identified: (i) +, the experimental data of Marie et al. (1997), (ii) **+**, predictions with the standard law of the wall for the κ-ϵ model, (iii) predictions with the wall conditions given by Eqs. (12) to (16), (iv) predictions with the wall conditions given by Eqs. (12) to (15) and the classical value of $C_{\epsilon 1}$ (that is, with the correction given by Eq. (16) turned off), (v) predictions with the wall formulation of Troshko and Hassan (2001), (vi) the linear near wall solution and (vii) the standard law of the wall.

The first striking notice is that the results for $\alpha_{g_{max}}$ = 0.016 and 0.068 are well below the classical law of the wall but above the numerical predictions. The three curves obtained with simulations based on the models of Bitencourt et al. (2008) and Troshko and Hassan (2001) are very close together. The inclusion or not of Eq. (16) in the formulation does not change much the results, differences are marginal.

The results for the intermediate void fraction $\alpha_{g_{max}}$ = 0.038, however, are very good. The vertical displacement of the logarithmic region is predicted almost exactly by both theories in most of the y^+ range.

Following the classical measurements of Reichardt (1939) and Klebanoff (1955), it has been a common practice in literature to consider $\sqrt{\overline{v^2}} \approx 0.4\,\sqrt{\overline{u^2}}$ so that for a two-dimensional flow $\kappa = 0.58\,\overline{u^2}$. An application of this consideration to the data of Marié et al. (1997) permits an assessment of κ_{Exp} in the fully turbulent region. Figures 4 and 5 compare the predictions given by the theories with the experiments.

For the lower void fractions, the results provided by the models that incorporate the effects of the local void fraction are much better. The evidence that a correction factor must be included in the lower boundary condition for κ is thus very strong. The peak value of $kappa$ is very well reproduced by the formualtion of Troshko and Hassan.

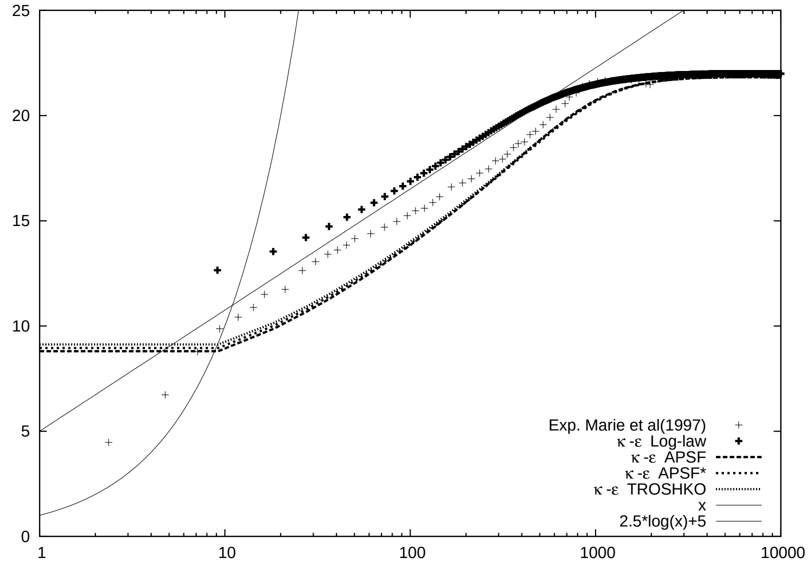

Figure 1. Mean velocity predictions for $\alpha_{g_{max}}$ = 0.016. APSF stands for the present formulation. The subscript is used in APSF to indicate that the corrections in $C_{\epsilon 1}$ due to void fraction effects are not accounted for.

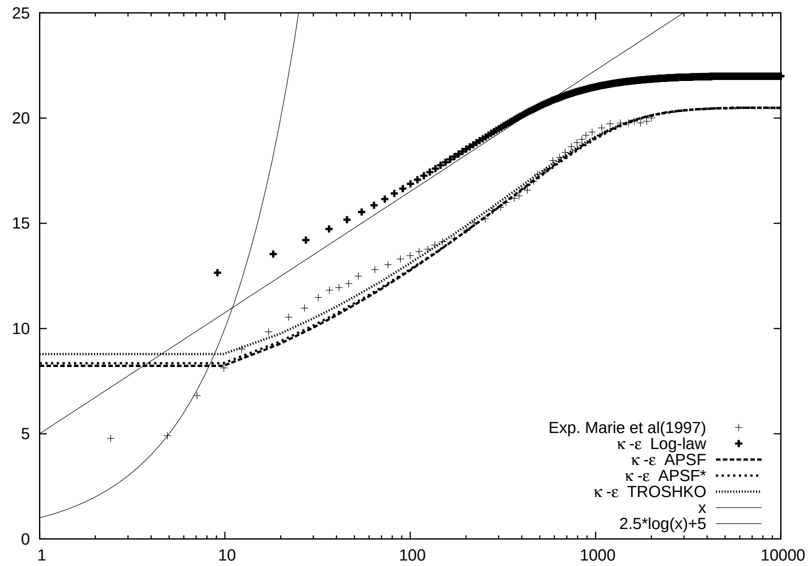

Figure 2. Mean velocity predictions for $\alpha_{g_{max}}$ = 0.038.

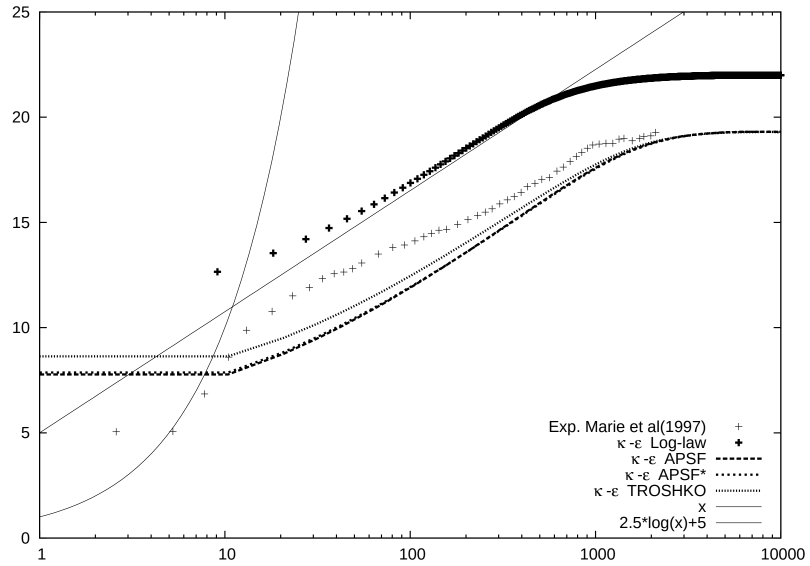

Figure 3. Mean velocity predictions for $\alpha_{g_{max}}$ = 0.068.

In relation to the higher void fractions, all models provide lower than expected peak values for κ. In the logarithmic region, as expected, all models give a nearly constant distribution.

The predictions for friction velocity are shown in Table 4. For the two lower void fractions, the model with $C_{\epsilon 1}$ gives the best agreement with the experimental data. Table 4 shows also that the results provided by the standard κ-ϵ model are 10% below the expected trend.

Table 4. Predictions of u_*.

$\alpha_{g_{max}}$	Present	Present without $C_{\epsilon 1}$ correction	Troshko and Hassan	LogLaw	Marie et al. (1997)
0.016	0.04798	0.04838	0.04882	0.04552	0.047
0.038	0.04919	0.04952	0.05043	0.04552	0.049
0.068	0.05057	0.05082	0.05216	0.04552	0.052

5. CONCLUSION

The present simulations have clearly shown that for two-phase flows the standard law of the wall has to be corrected to account for the near wall effects of the void fraction on the mean velocity profiles and friction coefficient. In particular, the functional dependence of the near wall mean velocity, kinetic energy and dissipation rate on the local void fraction as proposed by Bitencourt et al. (2008) and Troshko and Hassan (2001) have been tested against the data of Marie et al. (1997).

The present formulation, despite its simplicity, is show to model very well bubbly flow in turbulent boundary layers. The predictions obtained for the local friction velocity, in particular, are very good.

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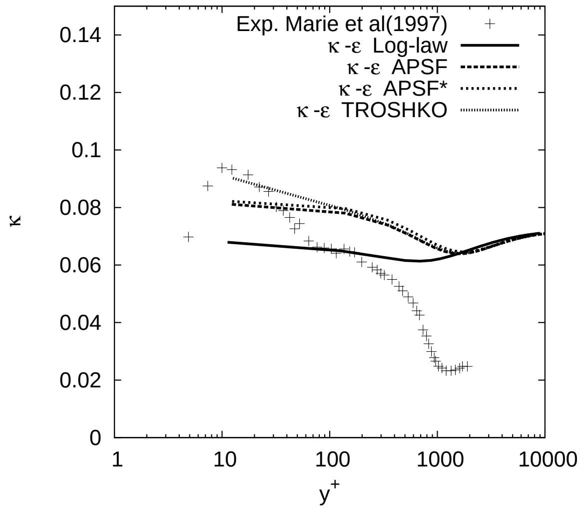

Figure 4. Turbulent kinetic energy predictions for $\alpha_{g_{max}}$ = 0.038.

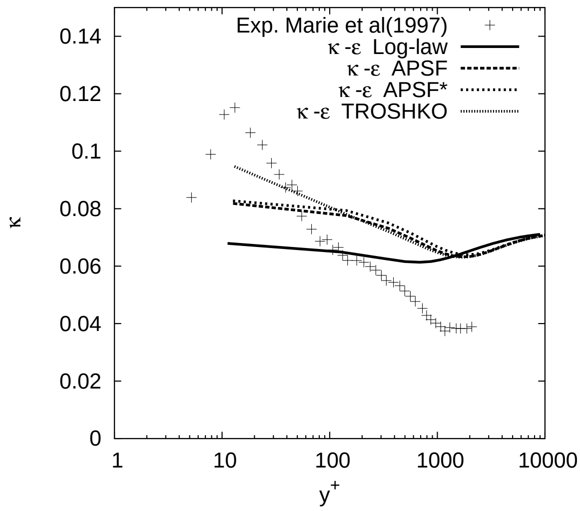

Figure 5. Turbulent kinetic energy predictions for $\alpha_{g_{max}}$ = 0.068.

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