GEOMETRY OPTIMIZATION OF STATICALLY INDETERMINATE TRUSSES

André Jacomel Torii, ajtorii@hotmail.com
Universidade Federal do Paraná

Abstract. *The problem of truss optimization has been studied, in most cases, considering a fixed geometry of the structure. This leads to the truss topology optimization problem, which only considers as design variables the truss members areas. However, the optimization of truss geometries can generally improve structural solutions, but at the cost of more complex problems. Some difficulties related to the optimization of truss geometries are: the truss geometry optimization problem is nonlinear by its nature; the evaluation of the gradients of the objective function and of the constraints can be difficult and sometimes approximated; and each possible geometry is in general associated to an optimum topology. However, the problem can be simplified if two assumptions are made. First, the areas of the truss members are considered as fixed values during the geometry optimization process, and then updated after this process is completed. The problem is then solved for the new members areas, and the procedure is repeated until the solution converges. Second, the objective function of the geometry optimization problem is related to the truss stiffness. Based on these two assumptions, analytical expressions for the gradients of the objective function and of the constraints are achieved. This improves the efficiency of the optimization method and reduces the computational effort of the numerical solution process. Moreover, since no assumptions are made on the topology being statically determinate, the proposed method can also be applied to statically indeterminate trusses.*

***Keywords**: truss optimization, structural optimization, geometry optimization, statically indeterminate*

1. INTRODUCTION

Methods for the optimization of truss topology, where the members areas are taken as design variable, are well established and there is a rich literature on this subject. However, the problem of truss geometry optimization, where the nodes positions are taken as design variables, was not studied as extensively as the topology one, and consequently references on this subject are not so vast. One of the main reasons why topology optimization was studied more frequently than geometry optimization is that the topology optimization problem can, in most cases, be stated as a linear programming one. For these problems, very efficient methods are available and it is possible to guarantee certain important mathematical properties of the solution (Arora, 2004; Nocedal and Wright, 1999; Rao, 1996). The truss geometry optimization problem is, instead, non linear by its nature, and therefore it needs to be solved by nonlinear optimization methods, that are in general more complex and computationally demanding than the linear programming techniques.

In the last decades the increase in computational resources made possible an increase in the use of nonlinear optimization methods, and thus the problem of geometry optimization was studied more extensively. However, some difficulties still hold. A first complication is that for every possible geometry there is an associated optimum topology. Thus, rigorously speaking, when solving a geometry optimization problem it is necessary to find the optimum topology associated with each geometry obtained during the optimization process. It can be noted, in this context, that while the optimization of truss topologies can be made without regarding its geometry, the inverse is, generally, not true. An optimum geometry depends on the topology considered. Thus, the concept of truss geometry optimization can be extended to the concept of *simultaneous optimization of truss topology and geometry* (Achtziger, 2006; Achtziger, 2007), indicating that optimum geometries are dependent on optimum topologies. Consequently, in order to be considered a non heuristic method, the problem must be solved simultaneously for the optimum members areas and nodes coordinates, which unfortunately leads to computational difficulties.

To avoid the coupling of geometry and topology optimization, a so called *alternating optimization* or *coordinate-wise optimization* has been proposed (Achtziger, 2006; Achtziger, 2007). In this approach, the geometry is fixed and the resulting topology optimization problem is solved. Then, the topology is fixed and the resulting geometry optimization problem is solved. These two steps are repeated, until the optimum solution is obtained. Since this process is expected to provide increasingly better solutions, it can be viewed as a descent method (Achtziger, 2006). It should be noted, however, that the uncoupling of geometry and topology optimization may lead to convergence difficulties, since in some design points it may be impossible to improve the solution by changing only the topology or only the geometry. It may happen that in these points, only simultaneous changes in both topology and geometry lead to improvements in the current solution, and thus the *alternating optimization* procedure may fail (Achtziger, 2006; Achtziger, 2007).

A second complication is that most nonlinear optimization methods need to evaluate not only the value of the objective function, but also its gradient. However, it is difficult to obtain analytical expressions for this gradient, since the relations between the nodes positions and the truss volume or the truss stiffness – two commonly used objective functions – are generally formulated in an implicit form. This difficulty can be avoided when considering that the optimum topology is always statically determinate for a single loading condition (Achtziger, 2006; Achtziger, 2007; Hemp, 1973; Pedersen, 1970). This leads to the conclusion that when an optimum geometry is obtained, the forces carried by the truss members do not depend on the members areas, but only on the current geometry of the truss. It is possible, in this ways, to obtain the members areas based on simple equilibrium equations and considering the members as fully stressed. Using these concepts, the optimization problem may consider as design variables only the nodes positions, while the members areas are just adapted to the current geometry by a fully stressed design. In this approach, the nodal coordinates are called *master* variables, since they are the design variables actually changed by the optimization algorithm. The members areas are called *follower* variables, since they are just adapted to a given geometry by a design procedure not directly related to the optimization algorithm. This approach is called *master-follower*, *implicit programming* or *nested* optimization, and has the advantage of updating both variables at each step of the solution process.

It is worth noting that *implicit programming* procedures for truss geometry optimization, as the one presented in Martínez et al. (2007), work only for statically determinate trusses. In fact, considering the optimum topology to be statically determinate simplifies the relationship between the nodal coordinates and the objective function, expressed for example in terms of truss stiffness or truss volume, making possible to obtain analytical expressions for the gradient of the objective function with respect to the design variables. However, statically indeterminate trusses are commonly found in practice. In all such cases, a geometry optimization method which relies on the assumption of statically determinate structures cannot be applied.

The purpose of this paper is to present a method for the simultaneous optimization of truss topology and geometry which can also be applied to design statically indeterminate optimal trusses (Torii, 2008). It is shown that by applying the concept of *alternating optimization* and using the truss stiffness as objective function, the problem of truss geometry optimization can be simplified to obtain analytical expressions for the gradient of the objective function. The use of the *alternating optimization* approach "uncouples" the problem in a topology optimization problem, with fixed geometry, and a geometry optimization problem, with fixed members areas. Therefore, in this approach two methods, one for topology optimization and one for geometry optimization, are required. The topology optimization method presented in (Hemp, 1973) is adopted. Since the problem of truss topology optimization is extensively described in literature (Achtziger, 2006; Achtziger, 2007; Hemp, 1973; Pedersen, 1970; Pedersen, 1993), this paper focuses on the geometry optimization for a fixed topology.

2. A METHOD FOR TRUSS GEOMETRY OPTIMIZATION

2.1 Formulation of the problem

The geometry optimization problem considers a fixed topology, which implies that the members areas are constant throughout the geometry optimization process. The geometry is optimized by maximizing the stiffness, or minimizing the work done by the external loads (Achtziger, 2006; Achtziger, 2007; Hemp, 1973; Pedersen, 1970). The problem is formulated as follows:

$$\min\ W(\mathbf{x}) = \mathbf{u}^T.\mathbf{F} \tag{1}$$

subjected to

$$g = \sum_{e=1}^{m} l_e.A_e - V_0 \leq 0\ , \tag{2}$$

$$\mathbf{K}.\mathbf{u} = \mathbf{F}\ , \tag{3}$$

$$lb \leq x_i \leq ub\ , \tag{4}$$

where $\mathbf{x}$ is the design vector composed of chosen nodal coordinates; $\mathbf{u}$ is the vector of nodal displacements; $\mathbf{F}$ is the vector of applied nodal forces; m is the number of elements; l_e is the length of element e; A_e is the area of element e; V_0 is the maximum volume of material to be used; g is the constraint on the volume of material; $\mathbf{K}$ is the stiffness matrix of the structure; lb and ub represent a lower bound and an upper bound, respectively, for the design variable x_i (nodal coordinate); W is the "work" done in the structure. Note that, for convenience, the quantity $W(\mathbf{x})$ will be called "work" done by the external loads, even if rigorously speaking it is twice the actual work. In fact, for optimization purposes, the division by two can be omitted, since the multiplication of the objective function by a constant does not change the optimum design vector (Arora, 2004).

Since it is known that maximizing the stiffness of the structure leads, for a single loading condition, to a decrease in the volume of material used (Achtziger, 2006; Achtziger, 2007; Hemp, 1973; Pedersen, 1970), the optimization method is expected to reduce the volume at each iteration. Consequently, the constraint of Eq. (2) may be dropped to simplify the problem and to improve the efficiency of the optimization algorithm. However, according to the classical approach found in literature (Achtziger, 2006; Achtziger, 2007; Hemp, 1973; Pedersen, 1970; Pedersen, 1993), the proposed optimization method is developed by considering also Eq. (2).

Regarding the other constraints, it is noted that the definition of bounds on the design variables by the use of Eq. (4) makes the problem better posed, since optimization problems with unbounded domains may present theoretical difficulties, and consequently burden the optimization algorithm (Arora, 2004; Nocedal and Wright, 1999). These bounds can be defined as upper and lower bounds in the nodal coordinates, which in two dimensions represent a rectangular region. Lower and upper bounds in the design variables can easily be incorporated in most optimization algorithms and are also known as box constraints. Besides, since these constraints are simple, they generally do not lead to a significant increase in computational effort when the optimization algorithm is applied.

The formulation defined by Eqs. (1)-(4) does not consider the structure as statically determinate, and thus it can also be applied for statically indeterminate trusses. Finally, after the optimum geometry is found, the topology is updated by truss topology optimization and the process repeated.

2.2. Evaluating the gradients – sensitivity analysis

The optimization problem can be solved more efficiently if the gradients of the objective function and of the constraint with respect to the design variables can be evaluated by analytical expressions. If such expressions are not available, finite differences can be used to approximate these gradients, but at the cost of a significant increase in computational effort and, possibly, reduction of the convergence rate of the procedure.

In order to obtain the gradient of the objective function and of the constraint, it is necessary to define its partial derivatives. Starting by the constraint of Eq. (2), the partial derivatives are

$$\frac{\partial g}{\partial x_i} = \sum_{e=1}^{m} \frac{\partial l_e}{\partial x_i} . A_e \ . \tag{5}$$

Note that since the topology is fixed, the areas A_i are not dependent on the design variables x_i. Since the partial derivatives of the element lengths $\partial l_e / \partial x_i$ can easily be computed, the evaluation of Eq. (5) and, consequently, of the gradient of the constraint, is simple. According to Fig. 1, the length of an element can be written as

$$l_e = \sqrt{(x_e^k - x_e^j)^2 + (y_e^k - y_e^j)^2} \ , \tag{6}$$

where x_e^j (x_e^k) and y_e^j (y_e^k) are the coordinates of the initial (end) node of the element e. The derivative of Eq. (6), after manipulation, gives:

$$\frac{\partial l_e}{\partial x_i} = \begin{cases} 0 & & \not\subset e \\ -\cos(\alpha) & & x_e^j \\ -\sin(\alpha) & \text{if } x_i = & y_e^j \\ \cos(\alpha) & & x_e^k \\ \sin(\alpha) & & y_e^k \end{cases} \ . \tag{7}$$

This derivative depends on whether the nodal coordinate x_i is an x or y coordinate of the initial or end node of the element. Besides, if the nodal coordinate x_i does not belong to the element e, the derivative is zero. Note that, for convenience, the design variables are called x_1, x_2, …, x_i, even for y coordinates.

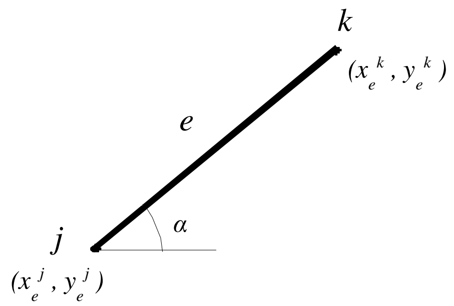


Figure 1. Cartesian coordinates x and y, initial node j, end node k, and inclination angle α, measured in the anti-clockwise direction, for the element e.

The partial derivatives of the objective function from Eq. (1), which are needed to obtain the gradient, can be written as

$$\frac{\partial W}{\partial x_i} = \frac{\partial \mathbf{u}^T}{\partial x_i}.\mathbf{F} + \mathbf{u}^T.\frac{\partial \mathbf{F}}{\partial x_i} = \mathbf{F}^T.\frac{\partial \mathbf{u}}{\partial x_i} + \mathbf{u}^T.\frac{\partial \mathbf{F}}{\partial x_i}.$$

Differentiating Eq. (3) gives:

$$\frac{\partial \mathbf{K}}{\partial x_i}.\mathbf{u} + \mathbf{K}.\frac{\partial \mathbf{u}}{\partial x_i} = \frac{\partial \mathbf{F}}{\partial x_i},$$

which, after rearrangement, is:

$$\frac{\partial \mathbf{u}}{\partial x_i} = \mathbf{K}^{-1}.\left(\frac{\partial \mathbf{F}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}.\mathbf{u}\right).$$

Substituting $\partial \mathbf{u}/\partial x_i$ back in the derivative $\partial W/\partial x_i$ gives:

$$\frac{\partial W}{\partial x_i} = \mathbf{F}^T.\mathbf{K}^{-1}.\left(\frac{\partial \mathbf{F}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}.\mathbf{u}\right) + \mathbf{u}^T.\frac{\partial \mathbf{F}}{\partial x_i}.$$

Since Eq. (3) can be written in transpose form as:

$$\mathbf{u}^T.\mathbf{K}^T = \mathbf{F}^T \;\therefore\; \mathbf{u}^T = \mathbf{F}^T.\mathbf{K}^{-1},$$

the expression for $\partial W/\partial x_i$ simplifies to

$$\frac{\partial W}{\partial x_i} = \mathbf{u}^T.\left(\frac{\partial \mathbf{F}}{\partial x_i} - \frac{\partial \mathbf{K}}{\partial x_i}.\mathbf{u}\right) + \mathbf{u}^T.\frac{\partial \mathbf{F}}{\partial x_i}$$

and then to

$$\frac{\partial W}{\partial x_i} = 2.\mathbf{u}^T.\frac{\partial \mathbf{F}}{\partial x_i} - \mathbf{u}^T.\frac{\partial \mathbf{K}}{\partial x_i}.\mathbf{u}, \tag{8}$$

Eq. (8) can be used to evaluate the gradient of the truss stiffness with the use of the nodal displacements **u**, the derivatives of the stiffness matrix $\partial\mathbf{K}/\partial x_i$, and the derivatives of the nodal forces $\partial\mathbf{F}/\partial x_i$. Since the nodal displacements **u** must be previously found in order to obtain the value of the objective function from Eq. (1), its use in Eq. (8) does not lead to a significant increase in computational effort. Only one system of linear equations has to be solved and this does not depend on the size of the problem. Moreover, no assumptions have been made on the truss being statically determinate or indeterminate, and the previous equations hold for both cases.

When **F** depends on **x**, the derivative $\partial\mathbf{F}/\partial x_i$ must be defined accordingly. This happens for example when the self weight of the structure is considered. However, in many cases the derivative $\partial\mathbf{F}/\partial x_\mathrm{i}$ can be taken as zero, since the applied nodal forces are independent of the nodal coordinates **x**. For these cases, Eq. (8) then gives:

$$\frac{\partial W}{\partial x_i} = -\mathbf{u}^T.\frac{\partial \mathbf{K}}{\partial x_i}.\mathbf{u} \tag{9}$$

Note that a similar equation was obtained in (Wang et al., 2002), for the evaluation of sensitivity numbers for the combined shape and topology optimization of truss structures.

Since the stiffness matrix of the entire structure can be written as the sum of all the element stiffness matrices, so does its derivative $\partial\mathbf{K}/\partial x_\mathrm{i}$. This summation of the individual matrices is the same as the assembling procedure, as described in (Bathe, 1996) and (Hutton, 2003). It can be formally expressed as follows:

$$\mathbf{K} = \sum_{e=1}^{m} \mathbf{k}_e \quad \therefore \quad \frac{\partial \mathbf{K}}{\partial x_\mathrm{i}} = \sum_{e=1}^{m} \frac{\partial \mathbf{k}_e}{\partial x_\mathrm{i}}, \tag{10}$$

where $\mathbf{k}_e$ is the stiffness matrix of the element e in global coordinates, and where the stiffness matrices of the individual elements are represented by lower case $\mathbf{k}$ in order to avoid confusion with the stiffness matrix of the whole structure $\mathbf{K}$. The stiffness matrix $\mathbf{k}$ of each element in global coordinates can be written as (Hutton, 2003):

$$\mathbf{k} = \mathbf{R}^T.\mathbf{k}_l.\mathbf{R} \ , \tag{11}$$

where $\mathbf{k}_l$ is the element stiffness matrix in the local coordinates and $\mathbf{R}$ is the coordinate transformation matrix, and the subscript e is dropped for formal convenience. The partial derivative of Eq. (11) is:

$$\frac{\partial \mathbf{k}}{\partial x_i} = 2.\mathbf{R}^T.\mathbf{k}_l.\frac{\partial \mathbf{R}}{\partial x_i} + \mathbf{R}^T.\frac{\partial \mathbf{k}_l}{\partial x_i}.\mathbf{R} \ . \tag{12}$$

Starting by the element stiffness matrix in local coordinates:

$$\mathbf{k}_l = \frac{A.E}{l}.\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \tag{13}$$

where A is the element area, E is the Young modulus of the material and l is the length of the element, the following derivatives of the stiffness matrix $\mathbf{k}_l$ are obtained:

$$\frac{\partial \mathbf{k}_l}{\partial x_i} = -\frac{A.E}{l^2}.\frac{\partial l}{\partial x_i}.\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}. \tag{14}$$

where the derivative $\partial l/\partial x_i$ is given by Eq. (7). Moreover, since the coordinate transformation matrix is (Hutton, 2003):

$$\mathbf{R} = \begin{bmatrix} \cos(\alpha) & \sin(\alpha) & 0 & 0 \\ 0 & 0 & \cos(\alpha) & \sin(\alpha) \end{bmatrix}, \tag{15}$$

its derivatives become:

$$\frac{\partial \mathbf{R}}{\partial x_i} = \begin{bmatrix} \dfrac{\partial \cos(\alpha)}{\partial x_i} & \dfrac{\partial \sin(\alpha)}{\partial x_i} & 0 & 0 \\ 0 & 0 & \dfrac{\partial \cos(\alpha)}{\partial x_i} & \dfrac{\partial \sin(\alpha)}{\partial x_i} \end{bmatrix}, \tag{16}$$

where, after some manipulation, the partial derivatives involved in Eq. (16) are:

$$\frac{\partial \cos(\alpha)}{\partial x_i} = \begin{cases} 0 & \text{if} \quad x_i \not\subset e \\ \dfrac{-1+\cos^2(\alpha)}{l} & \text{if} \quad x_i = x_e^j \\ \dfrac{\cos(\alpha).\sin(\alpha)}{l} & \text{if} \quad x_i = y_e^j \\ \dfrac{1-\cos^2(\alpha)}{l} & \text{if} \quad x_i = x_e^k \\ \dfrac{-\cos(\alpha).\sin(\alpha)}{l} & \text{if} \quad x_i = y_e^k \end{cases}, \quad \frac{\partial \sin(\alpha)}{\partial x_i} = \begin{cases} 0 & \text{if} \quad x_i \not\subset e \\ \dfrac{\cos(\alpha).\sin(\alpha)}{l} & \text{if} \quad x_i = x_e^j \\ \dfrac{-1+\sin^2(\alpha)}{l} & \text{if} \quad x_i = y_e^j \\ \dfrac{-\cos(\alpha).\sin(\alpha)}{l} & \text{if} \quad x_i = x_e^k \\ \dfrac{1-\sin^2(\alpha)}{l} & \text{if} \quad x_i = y_e^k \end{cases}. \tag{17}$$

This method for evaluating the objective function and its gradient requires that a matrix $\partial\mathbf{K}/\partial x_i$ is assembled for each design variables x_i. Consequently, many matrices $\partial\mathbf{k}/\partial x_i$ must be evaluated at each iteration. Even if these matrices can be easily evaluated, the number of times they are used makes the process time consuming. However, this approach has three main advantages. First, in order to evaluate the objective function and its derivatives it is necessary to solve only one system of linear equations. This is very attractive since this operation can become very cumbersome for large problems. Second, the derivatives evaluated by this method are analytical, improving in this way the efficiency of the optimization method (Arora, 2004; Nocedal and Wright, 1999; Rao, 1996). Third, the procedure is general and can be applied also for statically indeterminate trusses.

2.3. Alternative loading conditions

The method previously described can be modified in order to consider alternative loading conditions by taking the objective function as the sum of the "work" done by each load condition. Since the constraint from Eq. (2) represents the volume of the structure, it remains unchanged when considering alternative load conditions, and so does its derivatives. The only changes appear then in the objective function from Eq. (1) and its derivative from Eq. (9), which become

$$\min f(\mathbf{x}) = \sum_{\alpha=1}^{s} W_\alpha = \sum_{\alpha=1}^{s} \left(\mathbf{u}_\alpha{}^T . \mathbf{F}_\alpha \right) \tag{18}$$

and

$$\frac{\partial f}{\partial x_i} = \sum_{\alpha=1}^{s} \frac{\partial W_\alpha}{\partial x_i} = \sum_{\alpha=1}^{s} \left(-\mathbf{u}_\alpha{}^T . \frac{\partial \mathbf{K}}{\partial x_i} . \mathbf{u}_\alpha \right), \tag{19}$$

where α represents a particular loading condition, s is the number of loading conditions considered and f is now the objective function of the problem.

In order to obtain the displacements $\mathbf{u}_\alpha$ for each load condition it is necessary to solve the system of linear equations from Eq. (3), which now gives

$$\mathbf{K}.\mathbf{u}_\alpha = \mathbf{F}_\alpha, \tag{20}$$

for a given loading condition α.

From Eqs. (18) and (19) it can be seen that the new objective function (and consequently its derivative) is now the sum of the individual "works" done by each loading condition $\mathbf{F}_\alpha$ taken over its respective displacements $\mathbf{u}_\alpha$. The only change is that now the system of linear equations from Eq. (20) must be solved for each loading condition α.

It is important to note that the problem of maximization of the stiffness is mathematically the same as the minimization of volume only when only one loading condition is considered (Hemp, 1973). Thus, when using the geometry optimization method here proposed, there is no guarantee that the minimization of the work will actually lead to a minimization of the volume for every case. However, for most cases, the geometry optimization here proposed does leads to a reduction of the volume of the structure, as shown in following examples.

2.4. Some practical aspects

From Eqs. (7) and (17) it can be seen that the partial derivatives $\partial l/\partial x_i$, $\partial\cos(\alpha)/\partial x_i$ and $\partial\sin(\alpha)/\partial x_i$ are zero when the nodal coordinate x_i does not belong to the element e. In such cases, the matrix $\partial\mathbf{k}/\partial x_i$ from Eq. (12) becomes also a zero matrix. This states that when a given element e does not converge to a given nodal coordinate x_i, it does not contribute to the derivative of the stiffness matrix associated to this nodal coordinate, or:

$$\frac{\partial \mathbf{k}_e}{\partial x_i} = 0 \text{ if } x_i \not\subset e.$$

Therefore, instead of computing the matrix $\partial\mathbf{k}/\partial x_i$ for all the elements of the structure, it is more efficient to "remember" the elements that converge to each nodal coordinate and then compute $\partial\mathbf{k}/\partial x_i$ just for these elements. This can make the evaluation of $\partial\mathbf{K}/\partial x_i$ faster, since $\partial\mathbf{k}/\partial x_i$ must be evaluated for each element in order to assemble $\partial\mathbf{K}/\partial x_i$.

Considering for example the structure shown in Fig. 2.a. In order to evaluate $\partial\mathbf{K}/\partial x_7$ for the nodal coordinate x_7, it should be necessary to compute $\partial\mathbf{k}/\partial x_7$ for every element in the structure and then assemble $\partial\mathbf{K}/\partial x_7$. This would lead to the evaluation of eleven matrices $\partial\mathbf{k}/\partial x_7$, since there are eleven members in the structure. However, if the previous simplification is used, $\partial\mathbf{k}/\partial x_7$ is computed just for the elements that converge to the nodal coordinate x_7, and the matrix would be evaluated five times. Since $\partial\mathbf{K}/\partial x_i$ must be evaluated for each nodal coordinate x_i, the reduction in computational effort would be important.

Another important practical aspect is to define the nodal coordinates that have a fixed position in the optimization process. For the truss shown in Fig. 2.a, the nodal coordinates x_1, x_2, x_5, x_9 and x_{10} can be assumed as fixed, since they represent the nodes of supports and of applied forces. The most efficient way of considering fixed nodes in the optimization problem is not considering them as design variables. The design vector $\mathbf{x}$ is then composed of just the nodal coordinates which are free to move in the geometry optimization process. For the example of Fig. 2.a, the design vector would be

$$\mathbf{x}^T = [x_3 \quad x_4 \quad x_6 \quad x_7 \quad x_8 \quad x_{11} \quad x_{12}].$$

There are two different approaches for defining bounds on the design variables (nodal coordinates). In the first approach, bounds can be defined locally for each design variable, like shown in Fig. 2.b. In this case, there are different bounds for each design variable, and this can be accomplished by defining a "square" around each node. The second approach is that of defining bounds for all the design variables at once, like shown in Fig. 2.c. In this case, the bounds are the same for all design variables, and this is the same as defining a rectangle around the entire structure. These two approaches may lead to different results, mainly because the feasible domain defined in the first approach, that of Fig. 2.b, leads to a smaller feasible domain. Since the feasible domain is smaller, there is a lower number of possible solutions. However, note that the first approach may prevent the difficulty caused by node superposition (Achtziger, 2007), when the bounds are defined appropriately. Thus, this approach may be recommended when there are many nodes which are allowed to move in the optimization process.

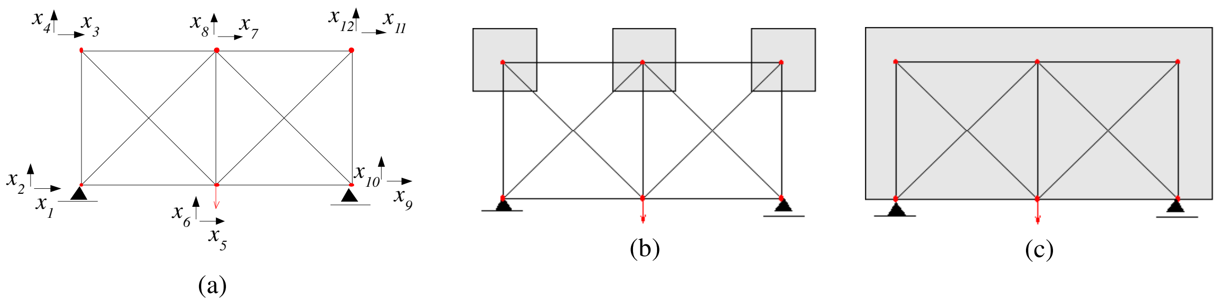


Figure 2. (a) Truss structure with nodal coordinates as design variables. The gray rectangles in (b) and (c) represent the feasible domains, bounded by box constraints on the nodal coordinates: bounds can be defined (b) locally for each design variable separately, and (c) globally for all the design variables at once.

2.5. Optimization algorithm

The algorithm for the *alternating* optimization of truss topology and geometry can be defined as follows:

1. Start the optimization process with $i = 0$, and an initial geometry $\mathbf{x}^0$;
2. Update the counter: $i = i+1$;
3. Fix the geometry $\mathbf{x}^{i-1}$ and find an optimum topology $\mathbf{A}^i$ (members areas) with volume V_0^i;
4. Fix the topology $\mathbf{A}^i$ and find an optimum geometry $\mathbf{x}^i$;
5. Check convergence on the volumes V_0^i and V_0^{i-1} or the design vectors $\mathbf{x}^i$ and $\mathbf{x}^{i-1}$ from the current and last iteration. If a convergence criterion is satisfied, stop the optimization process. If the solution does not converge, go to step 2.

As stated previously, when considering only one loading condition, the maximization of the truss stiffness is expected to lead to a reduction in its volume. Thus, the constraint of Eq. (2) may be dropped. In this case, the algorithm for the geometry optimization step is the following:

1. Start the geometry optimization process with: fixed members areas **A,** an initial geometry $\mathbf{x}^0$, lower and upper bounds on the design variables, and a counter $k = 0$;
2. Update the counter: $k = k+1$;
3. For each design variable x_i:
   a. For the members e that converge to x_i:
      i. Find the derivative of the stiffness matrix $\partial\mathbf{k}/\partial x_i$ for the member e using Eq. (12);
      ii. Assemble $\partial\mathbf{k}/\partial x_i$ to $\partial\mathbf{K}/\partial x_i$ by Eq. (10);
4. For each loading condition α:
   a. Solve the system of linear equations from Eq. (20) thus finding the vector of nodal displacements $\mathbf{u}_\alpha^k$ associated with the loading condition α;
   b. Obtain the value of the "work" W_α^k for the loading condition α from Eq. (1);
   c. For each design variable x_i:
      i. With the derivative of the stiffness matrix for the entire truss $\partial\mathbf{K}/\partial x_i$ find the value of $\partial W_\alpha/\partial x_i$ using Eq. (9);
   d. Compose the gradient of the "work" for the loading condition α with the values $\partial W_\alpha/\partial x_i$;
5. Sum the "works" W_α^k thus evaluating the objective function f^k from Eq. (18);
6. Sum the gradients of the "works" $\partial W_\alpha/\partial x_i$ thus evaluating the gradient of the objective function from Eq. (19);
7. Use f^k, grad(f^k), *lb* and *ub* to update the geometry to $\mathbf{x}^k$. This is made through the use of a nonlinear optimization method;

8. Check convergence on f or $\mathbf{x}$. If the problem converges, stop the algorithm. If the problem does not converge, go to step 2.

The choice of the nonlinear optimization method may modify the algorithm, since different methods are available. However, these are the steps for evaluating the information needed by most optimization methods (Arora, 2004; Nocedal and Wright, 1999; Rao, 1996).

3. NUMERICAL APPLICATIONS

The applications presented in Figs. 3-7 refer to the following parameters: Lx and Ly are the structure size in the x and y coordinate, respectively; Dx and Dy are the size of the admissible domain of the problem in the x and y coordinates, respectively; b is the size of box constraints applied locally to the design variables; F is the magnitude of the applied forces; A are the members areas; E is the Young modulus; σ is the allowable stress of the material; W is twice the work done in the structure by the external loads; and V is the volume of the structure. In Figs. 3-8 bars are represented by full lines, global bounds by dashed lines, nodes by circles, and forces by arrows (each arrow represents a force of magnitude F). Local bounds applied to nodal coordinates are not shown. The values of F, E, σ, and A do not affect the optimal geometry and are taken as unity.

The application shown in Fig. 3 is a simply supported "beam" with equal loads applied to the nodes of the lower chord (Fig. 3.a). The optimization problem is solved for two separate cases by assuming as design variables only the y coordinates of the upper chord nodes (Fig. 3.b), or both the x and y coordinates of such nodes (Fig. 3.c). In both cases the geometry optimization allowed to increase the stiffness of the structure. Clearly, allowing the nodes to move in both x and y directions leads to a better optimum geometry, since the feasible domain is larger. A similar application is considered in Fig. 4. This is a "beam" with a concentrated load (Fig. 4.a) and statically indeterminate support reactions. By assuming the y coordinates of the upper nodes as design variables, the geometry optimization allows to improve the truss geometry (Fig. 4.b).

The applications shown in Figs. 5 to 7 are dealing with simultaneous optimization of geometry and topology. These applications highlight that geometry optimization can contribute to reduce the volume of the structure when used together with topology optimization. Fig. 5 shows a common type of structure, where it is interesting to note how the geometry optimization procedure succeeded in reducing the volume of material used. The application shown in Fig.6 is a classical problem from structural optimization, and in this case note that the reduction in the volume of material used is not so significant, since the solution from just topology optimization is similar to classical solutions for this problem (Hemp, 1973). The example from Fig. 7 shows another case commonly found in engineering practice.

In Fig. 8, the solution from simultaneous optimization of geometry and topology when considering all loading conditions and only the most demanding loading condition are compared. In this case, note that considering all loading conditions leads to a different solution from when considering only the most demanding one.

It is important to note that since the convexity of the nonlinear geometry optimization problem is not studied here, there is no guarantee that the optimum solutions obtained are global minima. However, even if there is no proof that the solutions are global minima, it is demonstrated how the optimum design is generally much improved by the geometry optimization process.

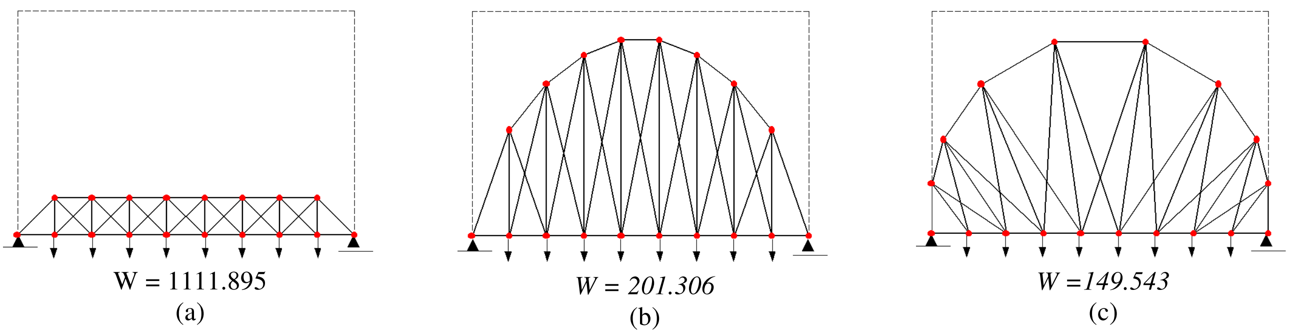
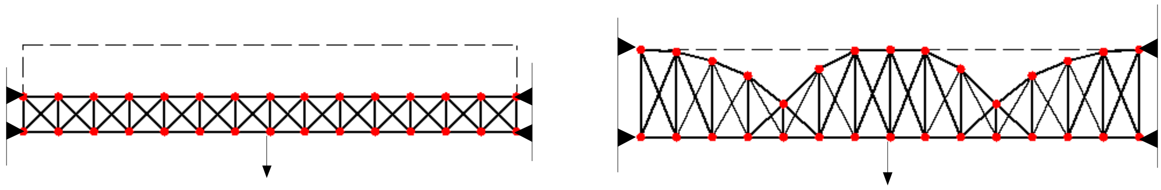


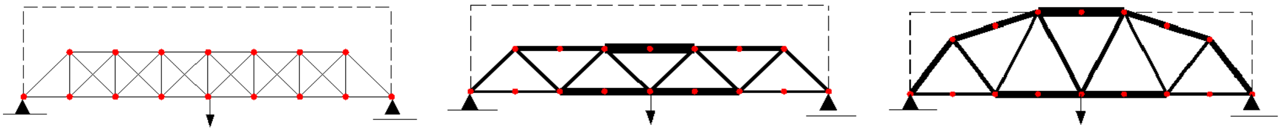
Figure 3. (a) Initial structure ($Lx = 9$, $Ly = 1$, $Dx = 9$, $Dy = 6$). Optimum geometry considering as design variables (b) the y coordinates only, and (c) both x and y coordinates of the upper chord nodes.



W = 33.151
(b)

W = 8.956
(c)

Figure 4. (a) Initial structure (*Lx = 14*, *Ly = 1, Dx = 14, Dy = 2.5*). (b) Optimum geometry considering the *y* coordinates of the upper chord nodes as design variables.

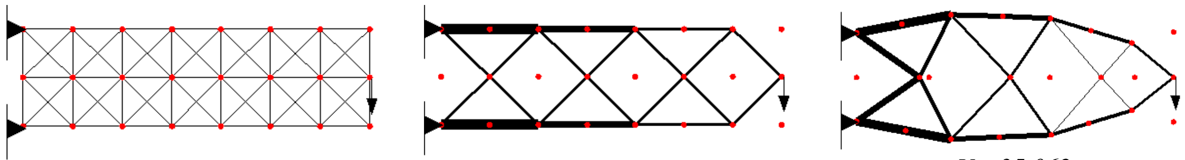


(a)

V = 24.000
(b)

V = 16.510 (c)

Figure 5. (a) Initial structure (*Lx = 8, Ly = 1*, *Dx = 8, Dy =2*). (b) Optimum topology. (c) Optimum design from simultaneous geometry and topology optimization, considering the *y* coordinates of the upper chord nodes as design variables.

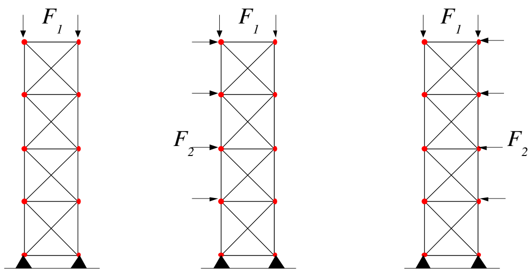


(a)

V= 38.000
(b)

V= 35.062
(c)

Figure 6. (a) Initial structure (*Lx = 7*, *Ly = 2*). (b) Optimum topology. (c) Optimum design from simultaneous geometry and topology optimization, considering as design variables the *x* and *y* coordinates of the upper and lower chord nodes, and the *x* coordinates of the nodes aligned with the applied load. Bounds on the design variables are defined locally, by a square of size *b =0.8* centered at the original node position.

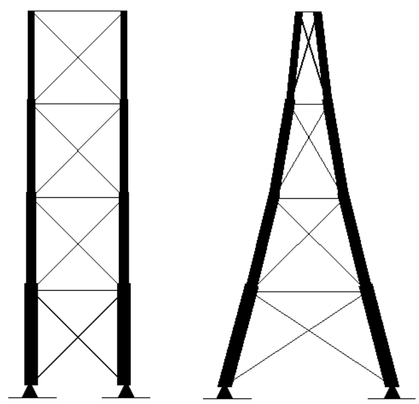


1

2

3

a)



V = 0.53
b)

V = 0.50
c)

Figure 7. a) Initial structure subjected to three alternative loading conditions (*Lx =1, Ly = 4*), considering F_1 = *10* and F_2 = *1*. Results from b)only topology optimization and c)simultaneous optimization of geometry and topology. In the geometry optimization step all nodes are allowed to “move” left and right. Bounds on the design variables for the geometry optimization are defined locally, by a square of size *b =0.4* centered at the original node position.

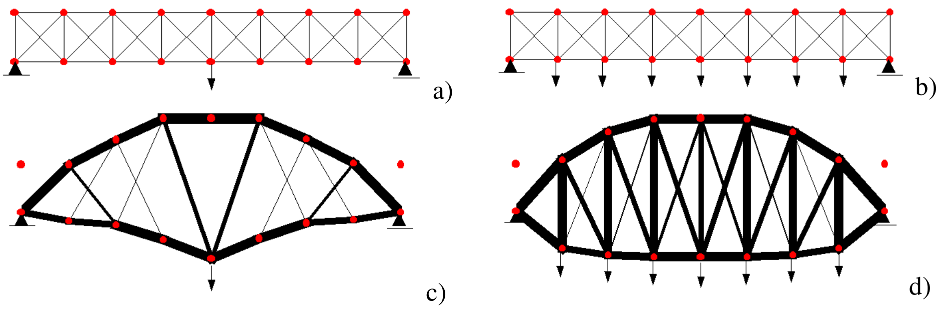


Figure 8. Initial structure (*Lx* =*8*, *Ly* = *1*) considering a) only loading condition at mid span and b) considering loading conditions at every node from the lower chord. Note that in b) each arrow represents a different loading condition, which are applied separately to the structure. Results considering c) only loading condition at mid span and d) all loading conditions. In the geometry optimization process all nodes (except the nodes of supports) are allowed to move up and down. Bounds on the design variables for the geometry optimization are defined locally, by a square of size *b* =*1* centered at the original node position.

4. CONCLUSIONS

A method for the simultaneous optimization of truss topology and geometry, which can also be applied to design statically indeterminate structures, has been presented. It has been shown that by applying the concept of alternating optimization and using the truss stiffness as objective function, the problem of truss geometry optimization can be simplified to obtain analytical expressions for the gradient of the objective function.

The results of the presented applications show that the proposed approach can lead to a significant reduction of the volume of the optimal structures. It is, however, worth noting that in this paper buckling is not considered. This is a crucial aspect in the design of truss structures, and consequently represents the most important developments required in the future to allow the application of the method to practical cases. Despite the need for these improvements, the proposed method has been proven to represent an efficient approach to simultaneous optimization of truss topology and geometry.

5. REFERENCES

Achtziger, W., 2006, “Simultaneous optimization of truss topology and geometry, revisited”, Martin P. Bendsøe et al. (eds), IUTAM Symposium on Topological Design Optimization of Structures, Machines and Materials: Status and Perspectives, 413–423, Springer.

Achtziger, W., 2007, “On simultaneous optimization of truss geometry and topology”, Structural and Multidisciplinary Optimization, vol. 33, Springer-Verlag.

Arora, J.S., 2004, “Introduction to optimum design”, 2nd Edition, Elsevier, San Diego, 728p.

Barbieri, E., Lombardi, M., 1998, “Minimum weight shape and size optimization of truss structures made of uncertain materials”, Structural Optimization, vol. 16, Springer-Verlag.

Bathe, K.J., 1996, “Finite element procedures”, Prentice Hall, New Jersey, 1037p.

Hagishita, T., Ohsaki, M., 2009, “Topology optimization of trusses by growing ground structure method”, Structural and Multidisciplinary Optimization, vol. 37, Springer-Verlag.

Hemp, W.S., 1973, “Optimum structures”, Clarendon Press, 123p.

Hutton, D.V., 2003, “Fundamentals of Finite Element Analysis”, McGraw-Hill, New York, 640p.

Martínez, P., Martí, P., Querin, O.M., 2007, “Growth method for size, topology, and geometry optimization of truss structures”, Structural and Multidisciplinary Optimization, vol. 33, Springer-Verlag.

Nocedal, J., Wright, S.J., 1999, “Numerical optimization”, Springer-Verlag, New York, 634p.

Pedersen, P., 1970, “On the minimum mass layout of trusses”, AGARD Conference Proceedings No. 36.

Pedersen, P., 1993, “Topology optimization of three-dimensional trusses”, In: M.P. Bendsoe, C.A. Mota Soares, (eds), Topology design of structures, Kluwer, Dordrecht.

Rao, S.S., 1996, “Engineering optimization: theory and practice”, 3rd Edition, John Wiley & Sons, New York, 922p.

Torii, A.J., 2008, “The role of self weight and loading conditions in optimization of truss structures”, Master Thesis, Master School F.lli Pesenti, Department of Structural Engineering, Politecnico di Milano, Italy.

Wang, D., Zhang, W.H., Jiang, J.S., 2002, “Combined shape and sizing optimization of truss structures”, Computational Mechanics, vol. 29, Springer-Verlag.

6. RESPONSIBILITY NOTICE

The author is the only responsible for the printed material included in this paper.