OUTPUT ONLY SYSTEM IDENTIFICATION OF A CANTILEVER STEEL BEAM

Letícia Fleck Fadel Miguel, letffm@ufrgs.br
Prof[a]. Dr[a]., Department of Mechanical Engineering, Federal University of Rio Grande do Sul, Porto Alegre, Brazil.

Leandro Fleck Fadel Miguel, leandro.miguel@ufrgs.br
Prof. Dr., Department of Civil Engineering, IPA University, Porto Alegre, Brazil.

Carlos Alberto Kern Thomas, carlos.thomas@ufrgs.br
Prof. M.Sc., Department of Mechanical Engineering, Federal University of Rio Grande do Sul, Porto Alegre, Brazil.

Abstract. *The wish for a better understanding of the dynamic behavior of systems, as well as the growing interest in areas such as structural health monitoring and design of vibration control devices, shows the importance of the use of robust and reliable methods to determine the dynamic properties of systems. One possible way to achieve that is to use output-only techniques. The present paper begins describing the theoretical bases of an efficient method for output-only system identification and an application example is demonstrated further. The studied system is a steel cantilever beam with a tip mass, which was instrumented and measured in the dependences of the Group of Applied Mechanics (GMAp) of the Federal University of Rio Grande do Sul (UFRGS). Through free vibration tests, in which were applied impact loads in the beam, its response is acquired. By using the Stochastic Subspace System Identification Method (SSI), the dynamic properties of the system are determined. Next, this beam is analyzed by finite element method and also through the theory of vibrations of continuous systems. Thus, the results obtained through the system identification method are compared with the numerical and theoretical results, showing that the dynamic properties obtained through system identification procedures are reliable.*

Keywords*: System Identification, Acquisition and Data Processing, Dynamic Analysis, Experimental Analysis, Numerical Analysis.*

1. INTRODUCTION

Growing demand in areas in which it is necessary the knowledge of the dynamic characteristics, such as structural health monitoring and design of vibration control devices, have been evidencing the relevance of the use of robust and reliable methods capable to determine the dynamic properties of systems, especially starting from the knowledge of output-only values.

A numerical approach, usually through finite element method, represents an alternative for obtaining the response of systems when these are subjected to any dynamic load. However, some theoretical concepts involved in the calculations incorporate certain simplifications, as in the case of the damping, for instance, that cannot reflect the real behavior of the system. Thus, experimental procedures play a fundamental role, because they evaluate the relation between theoretical predictions and these results. Some aspects of the system behavior cannot be predicted in the totality and they are only clear through dynamic tests. So, it is noticed that theoretical-numerical and experimental procedures are complemental for the description and understanding of the dynamic behavior of a system, not being able one to be substituted by the other. In that way, system identification is inside this context, because it has as objective the solution of the inverse problem, *i.e.*, the determination of a system that describes the relationship between an input and an output (Figure 1) or only a known output. In other words, system identification may be defined as the process in which is determined a mathematical model of a dynamic system starting from experimental data.

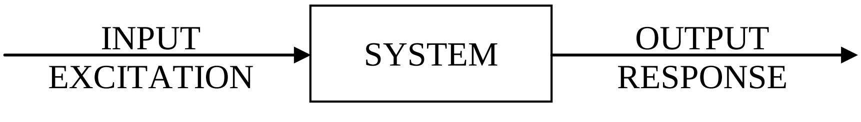

Figure 1. Representation of the relation between excitation and response of a system.

The modern data acquisition systems are facilitating studies of structural health monitoring and design of vibration control devices. For such studies, the system identification is also directly related, because the spectral properties of the structure should be obtained. Thus, dynamic tests should be carried out in order to obtain the modal parameters of a system. Some methods for the identification of dynamic properties of a system have been presented in the literature (Ewins, 1985; Juang, 1994; Maia and Silva, 1997). However, especially in cases of large structures, in which are not possible to employ standard testing equipments, the determination of the excitation may be difficult or even not viable. In those cases, it is practical and economical to use the excitation due to natural factors as wind, waves, or traffic, for instance, being called ambient excitation and the vibrations caused by those excitations called ambient vibrations. From the experimental point of view, this is the most practical method to excite and measure the dynamic parameters of many structures (Amani *et al.*, 2004). For this reason, methods that are capable to determine the dynamic properties starting only from response data of the system are extremely necessary. These situations, in which the system is excited by a not measured load and output-only measurements are available, have been referred as Stochastic System Identification.

In the last three decades, several researchers (Van Overschee and De Moor, 1993; Peeters and De Roeck, 1999) have been dedicating their efforts to the development of techniques able to produce reliable values of the dynamic properties of systems starting from output-only data. Among these, methods in the time domain stand out, because besides they are capable to identify same or very close natural frequencies, they identify a larger number of mode shapes than methods in the frequency domain (Inman, 1989). In contrast with these methods, which determine the modal data through Fourier transform, the methods in the time domain use a response values matrix, computing the dynamic characteristics numerically through the solution of an eigenvalue problem.

Within such context, in the present paper an experimental study is carried out acquiring response data of a cantilever steel beam with a tip mass that was instrumented and subjected to impact loads. These response data were used to determine the dynamic properties of the beam, obtained through the Stochastic Subspace System Identification Method (SSI) initially presented by Van Overschee and De Moor (1993) for applications in Electrical Engineering. This method, developed in the time domain, is based on the classic realization theory and it allows the determination of the modal characteristics through a model in state-space directly starting from the response data, without the need of previous calculation of covariances between the outputs or of the Markov parameters of the system. Peeters and De Roeck (1999), using Van Overschee and De Moor (1993) algorithms, introduced the concept of reference sensor in the formulation, allowing the reduction of the dimension matrices leading to a reduction of the computational cost. However, care should be taken with the selection of those reference sensors, because the quality of the results can be harmed. Finally, the natural frequencies obtained through SSI are compared with both numerical results obtained using a Finite Element Software (ANSYS - Version 10) and theoretical results obtained using the vibration theory of continuous systems.

2. STOCHASTIC SUBSPACE SYSTEM IDENTIFICATION (SSI)

The identification method described starts from a discrete-time state-space model given in Equation (1):

$$\begin{aligned}\vec{x}(k+1) &= \boldsymbol{A}\vec{x}(k)+\vec{w}(k)\\ \vec{y}(k) &= \boldsymbol{C}\vec{x}(k)+\vec{v}(k)\end{aligned} \tag{1}$$

in which k is the discrete time instant and $t = k\,\Delta t$, $\vec{x}(k)$ is the state vector, $\vec{y}(k)$ is the output vector, $\boldsymbol{A}$ is the state matrix and $\boldsymbol{C}$ is the output matrix. The vectors $\vec{w}(k)$ and $\vec{v}(k)$ represent respectively the noise due to the disturbances and modeling inaccuracies and the measurement noise due to the system acquisition inaccuracies. They are both not measurable vector signals assumed to be zero mean, modeled as Gaussian white noise and with covariances matrices presented in Equation (2):

$$E\left[\begin{pmatrix}\vec{w}_p\\ \vec{v}_p\end{pmatrix}\begin{pmatrix}\vec{w}_q^T & \vec{v}_q^T\end{pmatrix}\right]=\begin{pmatrix}\boldsymbol{Q} & \boldsymbol{S}\\ \boldsymbol{S}^T & \boldsymbol{R}\end{pmatrix}\delta_{pq} \tag{2}$$

in which $E[]$ is the expected value operator, δ_{pq} is the Kronecker delta and p, q are two arbitrary time instants.

Thus, in stochastic representation, the input is now implicitly modeled by the noise terms considering that it behaves as a stationary white noise process with zero mean. The main property of those systems indicates that the output covariances can be considered as Markov parameters of the deterministic linear time-invariant system, constituting the solution to the stochastic identification problem: the output covariance sequence can be estimated from the measurement data; so if the estimated output covariance sequence can be decomposed in a similar way, the state-space matrices are found. Starting from this idea some identification methods were proposed.

However, due to its formulation, the Stochastic Subspace System Identification Method (SSI) avoids this previous computation of covariances between the outputs. It is replaced by projecting the row space of future outputs into the row space of past outputs. The idea behind this projection, which apply robust numerical techniques such as QR factorization, is to retain in the past all the information that is useful to predict the future.

It is useful in the development of the SSI method to gather the output measurements in a block Hankel matrix with $2i$ block rows and N columns, in which N is the number of time samples. The first i blocks have r rows, the last i have l rows. The Hankel matrix can be divided into a past reference and a future part, given in Equation (3):

$$\boldsymbol{H}^{ref} = \frac{1}{\sqrt{N}} \left[\begin{array}{cccc} y_0^{ref} & y_1^{ref} & \cdots & y_{N-1}^{ref} \\ y_1^{ref} & y_2^{ref} & \cdots & y_N^{ref} \\ \vdots & \vdots & \ddots & \vdots \\ y_{i-1}^{ref} & y_i^{ref} & \cdots & y_{i+N-2}^{ref} \\ \hline y_i & y_{i+1} & \cdots & y_{i+N-1} \\ y_{i+1} & y_{i+2} & \cdots & y_{i+N} \\ \vdots & \vdots & \ddots & \vdots \\ y_{2i-1} & y_{2i} & \cdots & y_{2i+N-2} \end{array} \right] = \left[\frac{\boldsymbol{Y}_{0|i-1}^{ref}}{\boldsymbol{Y}_{i|2i-1}} \right] = \left[\frac{\boldsymbol{Y}_p^{ref}}{\boldsymbol{Y}_f} \right] \begin{array}{l} \updownarrow ri \\ \updownarrow li \end{array} \tag{3}$$

Subscripts p and f refer to the past and future, respectively. Another division is obtained by adding one block row to the past references and omitting the first block row of the future outputs, according to Equation (4):

$$\boldsymbol{H}^{ref} = \left[\begin{array}{c} \boldsymbol{Y}_{0|i}^{ref} \\ \hline \boldsymbol{Y}_{i|i} \\ \hline \boldsymbol{Y}_{i+1|2i-1} \end{array} \right] = \left[\begin{array}{c} \boldsymbol{Y}_p^{ref+} \\ \hline \boldsymbol{Y}_{i|i}^{\sim ref} \\ \hline \boldsymbol{Y}_f^{-} \end{array} \right] \begin{array}{l} \updownarrow r(i+1) \\ \updownarrow \; l-r \\ \updownarrow l(i-1) \end{array} \tag{4}$$

The projections play an important role in the stochastic subspace system identification. The notation and definition of this projection is defined in Equation (5):

$$\boldsymbol{P}_i^{ref} \equiv \boldsymbol{Y}_f / \boldsymbol{Y}_p^{ref} \equiv \boldsymbol{Y}_f (\boldsymbol{Y}_p^{ref})^T (\boldsymbol{Y}_p^{ref} (\boldsymbol{Y}_p^{ref})^T)^\dagger \boldsymbol{Y}_p^{ref} \tag{5}$$

in which $(\circ)^\dagger$ denotes the Moore-Penrose pseudo-inverse of a matrix.

Introducing the robust numerical QR factorization in the Hankel matrix Equation (3) into Equation (5) yields a very simple expression for the projections $\boldsymbol{P}_i^{ref}$, presented in Equation (6):

$$\boldsymbol{P}_i^{ref} = \begin{bmatrix} \boldsymbol{R}_{21} \\ \boldsymbol{R}_{31} \\ \boldsymbol{R}_{41} \end{bmatrix} \boldsymbol{Q}_1^T \tag{6}$$

The main theorem of stochastic subspace identification states that the projection $\boldsymbol{P}_i^{ref}$ can be factorized as the product of the extended observability matrix $\boldsymbol{O}_i$ and the Kalman filter state sequence $\widehat{\boldsymbol{X}}_i$, according to Equation (7):

$$\boldsymbol{P}_i^{ref} = \boldsymbol{O}_i \widehat{\boldsymbol{X}}_i \equiv \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{CA} \\ \cdots\cdot \\ \boldsymbol{CA}^{i-l} \end{bmatrix} \begin{bmatrix} \widehat{\bar{x}}_i & \widehat{\bar{x}}_{i+1} & \ldots & \widehat{\bar{x}}_{i+N-1} \end{bmatrix} \updownarrow n \tag{7}$$

The projection matrix has rank equals to n because it is the product of a matrix with n columns and a matrix with n rows [Equation (7)]. A reliable tool to numerically evaluate the rank of a matrix is the singular value decomposition (SVD). After omitting the zero singular values and corresponding singular vectors, the application of the SVD to the projection matrix can be carried out such that the extended observability matrix and the Kalman filter state sequence are obtained by splitting this decomposition in two parts, as shown in Equation (8):

$$\begin{aligned} \boldsymbol{P}_i^{ref} &= \boldsymbol{U}_1 \boldsymbol{S}_1 \boldsymbol{V}_1^T \\ \boldsymbol{O}_i &= \boldsymbol{U}_1 \boldsymbol{S}_1^{1/2} \\ \hat{\boldsymbol{X}}_i &= \boldsymbol{O}_i^{\dagger} \boldsymbol{P}_i^{ref} \end{aligned} \tag{8}$$

in which $\boldsymbol{U}_1 \in R^{li\ x\ n}$ and $\boldsymbol{V}_1 \in R^{N\ x\ n}$ are orthonormal matrices and $\boldsymbol{S}_1 \in \left(R_0^+\right)^{n\ x\ n}$ is a diagonal matrix containing the positive singular values in descending order. The order of the system is the number of non-zero singular values.

In order to obtain the system matrices, two algorithms can be used:
a) Algorithm 1: Using the states
Using the Hankel matrix, another projection can be defined, as shown in Equation (9):

$$\boldsymbol{P}_{i-1}^{ref} \equiv \boldsymbol{Y}_f^- / \boldsymbol{Y}_p^{ref+} = \begin{bmatrix} \boldsymbol{R}_{41} & \boldsymbol{R}_{42} \end{bmatrix} \begin{bmatrix} \boldsymbol{Q}_1^T \\ \boldsymbol{Q}_2^T \end{bmatrix} = \boldsymbol{O}_{i-1} \hat{\boldsymbol{X}}_{i+1} \tag{9}$$

The extended observability matrix $\boldsymbol{O}_{i-1}$ is obtained after rejecting the last *l* rows of $\boldsymbol{O}_i$, and the state sequence $\hat{\boldsymbol{X}}_{i+1}$ is computed according to Equation (10):

$$\hat{\boldsymbol{X}}_{i+1} = \boldsymbol{O}_{i-1}^{\dagger} \boldsymbol{P}_{i-1}^{ref} \tag{10}$$

Finally, the system matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ can be evaluated from the linear equation system of Equation (11):

$$\begin{bmatrix} \boldsymbol{A} \\ \boldsymbol{C} \end{bmatrix} = \begin{bmatrix} \hat{\boldsymbol{X}}_{i+1} \\ \boldsymbol{Y}_{i|i} \end{bmatrix} \hat{\boldsymbol{X}}_i^{\dagger} \tag{11}$$

b) Algorithm 2: Using the extended matrices
The matrices $\boldsymbol{A}$ and $\boldsymbol{C}$ can be obtained directly starting from the extended observability matrix presented in Equation (8).

Defining the matrix $\underline{\boldsymbol{O}_i} = \boldsymbol{O}_{i-1}$, obtained discarding the last *l* lines of $\boldsymbol{O}_i$ and the matrix $\overline{\boldsymbol{O}_i}$ obtained discarding the first *l* lines of $\boldsymbol{O}_i$, Equation (12) is obtained:

$$\boldsymbol{A} = \underline{\boldsymbol{O}_i}^{\dagger} \overline{\boldsymbol{O}_i} \tag{12}$$

The matrix $\boldsymbol{C}$ is simply the first *l* lines of $\boldsymbol{O}_i$.

After obtaining the system matrices they should be used for a modal analysis of the structure. The dynamic behavior is characterized by their eigenvalues and eigenvectors through its transformation for the continuous time, as shown in Equation (13):

$$\boldsymbol{A} = \boldsymbol{\Psi \Lambda \Psi}^{-1}, \quad \boldsymbol{\Lambda} = diag(\lambda_q) \in C^{n \times n}, \quad \boldsymbol{\Psi}_c = \boldsymbol{\Psi}, \quad \lambda = \frac{ln(\lambda_q)}{\Delta t} \tag{13}$$

in which $\Lambda = diag(\lambda_q)$ is a diagonal matrix containing the discrete-time eigenvalues. The eigenvalues occur in complex conjugated pairs and they can be written as Equation (14):

$$\lambda_{c_q}, \lambda_{c_q}^* = -\xi_q \omega_q \pm j \omega_q \sqrt{1 - \xi_q^2} \tag{14}$$

in which ξ_q is the damping ratio of the mode *q* and ω_q is the corresponding natural frequency to that mode.
The mode shapes at sensor locations $\boldsymbol{\Phi}_q$ are the observed parts of the eigenvectors of the system $\boldsymbol{\Psi}$ and they are obtained as shown in Equation (15):

$$\boldsymbol{\Phi} = \boldsymbol{C \Psi} \tag{15}$$

Thus, the modal parameters ω_q, ξ_q and $\boldsymbol{\Phi}_q$, are obtained from the identified system matrices, $\boldsymbol{A}$ and $\boldsymbol{C}$.

There are several variants of the stochastic subspace identification method, differing by the multiplication of a weighting function in the projection matrix $\boldsymbol{P}_i^{ref}$, before the singular value decomposition, determining the state-space basis in which the model will be identified, as presented in Equation (16):

$$\boldsymbol{W}_1 \boldsymbol{P}_i^{ref} \boldsymbol{W}_2 = \boldsymbol{U}_1 \boldsymbol{S}_1 \boldsymbol{V}_1^T \tag{16}$$

Three versions or variants (Van Overschee and De Moor, 1993) of the Stochastic Subspace System Identification Method (SSI) are presented: PC (principal component), UPC (unweighted principal component) and CVA (canonical variate algorithm). Table 1 shows the weighting functions of these variants, in which $\boldsymbol{I}$ is the identity matrix.

Table 1. Weighting functions of the SSI variants.

	$\boldsymbol{W_1}$	$\boldsymbol{W_2}$
PC	$\boldsymbol{I}$	$\boldsymbol{Y}_p^{ref} \left[\boldsymbol{Y}_p^{ref} \boldsymbol{Y}_p^{ref^T} \right]^{-1/2} \boldsymbol{Y}_p^{ref}$
UPC	$\boldsymbol{I}$	$\boldsymbol{I}$
CVA	$\left[\boldsymbol{Y}_f \boldsymbol{Y}_f^T \right]^{-1/2}$	$\boldsymbol{I}$

According to results obtained in previous papers (Fadel Miguel and Menezes, 2006; Fadel Miguel *et al.*, 2006 and Fadel Miguel, 2007), in which a comparison among the two algorithms and the three versions was made, in the present paper the authors opted to use the second algorithm, which uses the extended matrices, and the CVA version.

3. EXAMPLE: CANTILEVER STEEL BEAM WITH A TIP MASS

In order to demonstrate the efficiency of the Stochastic Subspace System Identification Method (SSI) described in the previous section, experimental dynamic tests in a cantilever steel beam with a tip mass, which was later identified using SSI, were carried out in the dependences of the Group of Applied Mechanics (GMAp) of the Federal University of Rio Grande do Sul (UFRGS). Next, the results of the first five natural frequencies obtained with SSI are compared with the frequencies obtained using the finite element software ANSYS. Besides, these values are also compared with the theoretical natural frequencies obtained using the continuous system vibration expressions. It should be emphasized that SSI also allows the identification of the damping ratios of the system, which is only possible with an experimental modal analysis.

More examples demonstrating the efficiency of the SSI, including the determination of the mode shapes, may be found in Fadel Miguel and Menezes (2006), Fadel Miguel *et al.* (2006) and Fadel Miguel (2007). The use of SSI as a first step for structural damage detection may be found in Fadel Miguel *et al.* (2007), for instance.

3.1. Experimental Analysis

The studied system is a cantilever steel beam of rectangular section with a tip mass. The material and geometrical properties of the tested beam are presented in Table 2.

Table 2. Material and geometrical properties of the tested beam.

Free length (*L*)	**Base (*b*)**	**Height (*h*)**	**Density (ρ)**	**Young's modulus (*E*)**
47cm	2.5cm	0.115cm	7618.5kg/m^3	197.19GPa

The following equipments, shown in Figure 2, were used to acquire the dynamic response of the system: a piezoelectric accelerometer of Brüel & Kjaer, type 4332, with mass of 30.2 grams and more 0.2 grams of its support; a signal conditioning / amplifier; an acquisition board PC - Card DAS 16/330 of ComputerBoards and a notebook with the software Matlab - version 7, with the toolbox for data acquisition.

Figure 2. Instrumented beam and equipments used for data acquisition.

Fifteen free vibration tests were carried out with this beam with a tip mass (which is composed by the accelerometer and its support mass). In some of them just one impact load was applied. In each test, a different excitation point was chosen in order to increase the deformation energy of other modes besides the fundamental. In other tests a sequence of impact loads was applied, also in different points of the beam. Examples of responses obtained for each one of these cases are shown in Figure 3. The responses were obtained using an acquisition rate of 2000Hz during 10s and 15s and also using an acquisition rate of 1000Hz during 20s.

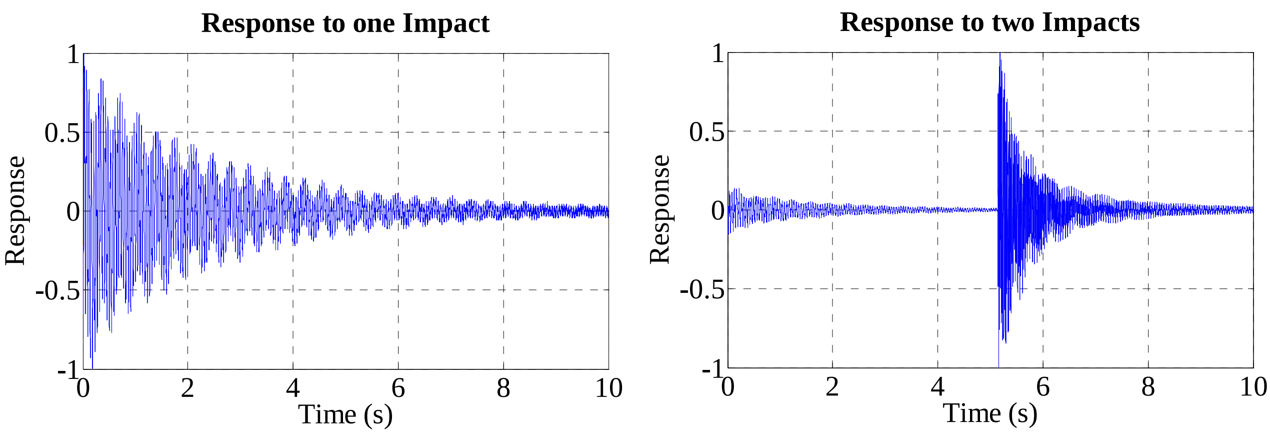

Figure 3. Response of the beam subjected to impact loads.

In order to reduce the number of data and to turn the identification more accurate in the ranges of frequencies of interest, the output data are filtered with an eight-order Chebyshev type I lowpass filter (Peeters and De Roeck, 1999). Next, the identification is carried out using the Stochastic Subspace System Identification Method (SSI) presented in Section 2. In this paper the authors chosen to use the second algorithm, which uses the extended matrices, and the CVA variant. For the selection of stable poles, stabilization diagrams were obtained, as may be seen in Figure 4.

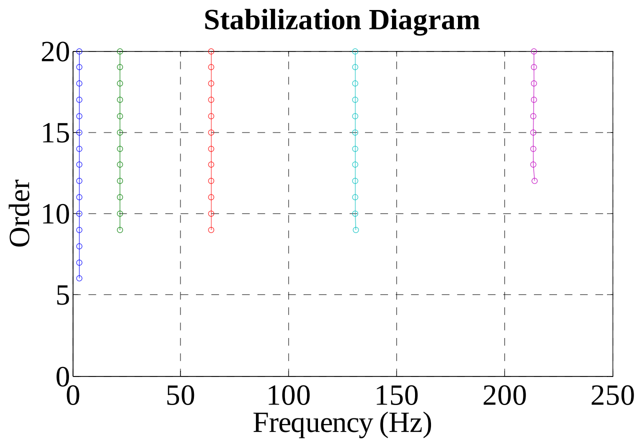

Figure 4. Stabilization diagram.

The fifteen response signals obtained in each one of the tests were used in the system dynamic properties identification. Table 3 shows the obtained mean values of all identifications with each one of the signals for the first five natural frequencies and damping ratios and their respective coefficients of variation.

Table 3. Mean values of the first five natural frequencies and damping ratios identified with SSI and their respective coefficients of variation.

Mode	Natural Frequency (Hz)	Coefficient of Variation of the Natural Frequency (%)	Damping Ratio	Coefficient of Variation of the Damping Ratio (%)
1st	2.86	0.58	0.0205	12.71
2nd	22.04	0.33	0.0047	43.61
3rd	64.75	0.30	0.0068	48.36
4th	131.68	0.10	0.0024	37.14
5th	214.62	0.50	0.0076	55.91

As it can be observed in Table 3, the coefficients of variation of the natural frequencies are relatively low, suggesting that the method converges for the real frequencies of the system. Moreover, as already expected, these coefficients of variation are much smaller than the obtained for the damping ratio, because the values of damping are very low. In spite of the values appears to be high, they are inside an acceptable range, as it is known that damping is the most difficulty dynamic parameter to be quantified.

3.2. Numerical Analysis

In order to compare the natural frequencies obtained using the SSI with numerical calculations, at this stage, a numerical modal analysis of the system was performed using the ANSYS software - version 10. The system was discretized in 47 beam elements and one more mass element to take into account the mass of the accelerometer and of its support (tip mass).

The values of the first five natural frequencies obtained with the ANSYS software may be seen in Table 4. A comparison between these results and the ones using the SSI method can be seem later in Table 6.

Table 4. First five natural frequencies obtained with ANSYS software.

Mode	Natural Frequency (Hz)
1st	2.88
2nd	21.42
3rd	64.07
4th	130.35
5th	220.55

3.3. Theoretical Analysis

The determination, in an analytical way, of the natural frequencies of transverse vibration of a beam can be made using the vibration theory of continuous systems. In this case, the differential equation of the motion is expressed by Equation (17):

$$\frac{\partial^2 w(x,t)}{\partial t^2} + c^2 \frac{\partial^4 w(x,t)}{\partial x^4} = 0 \,, \quad c = \sqrt{\frac{EI}{\rho A}} \tag{17}$$

in which *w(x,t)* is the displacement in the transverse direction to the beam axis of the mean line of the section *x* in the time *t*; *t* is the time; *x* is the position in the beam (from 0 until the total length *L*); *E* is the Young's modulus; *I* is the inertia moment of the cross section; ρ is the density and *A* is the area of the cross section.

The variable separation method is used to solve this differential equation. Thus, the solution may be expressed by Equation (18):

$$w(x,t) = X(x)T(t) \tag{18}$$

Substituting Equation (18) in (17) and solving the two resulting differential equations, the expressions (19) and (20) are obtained:

$$T(t) = a \sin \omega_n t + b \cos \omega_n t \tag{19}$$

$$X(x) = C_1 \cos \beta x + C_2 \sin \beta x + C_3 \cosh \beta x + C_4 \sinh \beta x \tag{20}$$

In which *a*, *b*, C_1, C_2, C_3 and C_4 are constants that are determined from two initial conditions and four boundary conditions and β is presented in Equation (21):

$$\beta^4 = \frac{\omega_n^2}{c^2} = \frac{\rho A \omega_n^2}{EI} \tag{21}$$

Therefore, the natural frequencies ω_n are expressed by Equation (22):

$$\omega_n = \beta^2 c = \beta^2 \sqrt{\frac{EI}{\rho A}} = (\beta L)^2 \sqrt{\frac{EI}{\rho A L^4}} \tag{22}$$

Because the differential equation contains partial derivate of fourth order, it requests four boundary conditions (two in each extremity) to determine natural frequencies and mode shapes.

In the case of the cantilever beam with tip mass in study, taking into account the mass of the accelerometer *m* present in the free extremity, the boundary conditions are:

- Fixed extremity:
  - displacement = $w = 0$
  - rotation $= \frac{\partial w}{\partial x} = 0$
- Free extremity with concentrated mass:
  - bending moment $= EI \frac{\partial^2 w}{\partial x^2} = 0$
  - shear $= EI \frac{\partial^3 w}{\partial x^3} = -m \frac{\partial^2 w}{\partial t^2}$

in which *m* is the mass of the accelerometer plus its support, equal to 0.0304kg (tip mass).

Deriving Equation (20) and substituting the boundary conditions, Equation (23) is obtained:

$$\cos(\beta L)\cosh(\beta L) + \frac{m}{\rho A L} \beta L \cos(\beta L)\sinh(\beta L) - \frac{m}{\rho A L} \beta L \sin(\beta L)\cosh(\beta L) = -1 \tag{23}$$

Solving Equation (23) for the first five mode shapes, it is obtained:

$\beta_1 L$ = 1.53948946
$\beta_2 L$ = 4.19510184
$\beta_3 L$ = 7.25594416
$\beta_4 L$ = 10.34984186
$\beta_5 L$ = 13.46302290

Substituting these values in Equation (22) and dividing the results by 2π, the first five natural frequencies (in Hz) of transversal vibration of the beam, taking into account the additional mass in the free extremity caused by the accelerometer and its support (tip mass), are obtained. These frequencies are presented in Table 5 and they will be compared with the ones obtained using the SSI method in Table 6.

Table 5. First five natural frequencies obtained using the vibration theory of continuous systems.

Mode	Natural Frequency (Hz)
1st	2.88
2nd	21.42
3rd	64.07
4th	130.35
5th	220.56

3.4. Comparison of the Results

Table 6 presents a comparison of the results of the first five natural frequencies obtained through SSI, ANSYS and the vibration theory of continuous systems.

Table 6. Comparison among the results of the first five natural frequencies.

Mode	SSI (Hz)	Numerical (Hz)	Theoretical (Hz)	Difference: SSI - Numerical (%)	Difference: SSI - Theoretical (%)
1st	2.86	2.88	2.88	0.70	0.70
2nd	22.04	21.42	21.42	2.81	2.81
3rd	64.75	64.07	64.07	1.05	1.05
4th	131.68	130.35	130.35	1.01	1.01
5th	214.62	220.55	220.56	2.76	2.77

As it may be observed in Table 6, the numerical and theoretical results are practically identical and they are also very close to the results obtained by the system identification method, showing a maximum difference of 2.81%. It should be pointed out that these values are the average identified frequencies for 15 response signals. Moreover, some responses identified certain modes better than others.

4. CONCLUSIONS

This paper presented the use of the Stochastic Subspace System Identification Method (SSI) for the determination of the dynamic parameters of a cantilever steel beam with a tip mass, starting from the knowledge of only output data of the system obtained experimentally through impact loads applied to the beam.

The results obtained through SSI were compared with both the results obtained numerically using the finite element software ANSYS - version 10 and the analytical results obtained using the vibration theory of continuous systems.

As it could be observed, the first five natural frequencies obtained using the SSI are very close to the ones obtained using the numerical and theoretical methods. The maximum difference was only 2.81% and the maximum coefficient of variation was 0.58%, showing that the SSI is really very efficient in determining the natural frequencies of the system. The maximum damping ratio variation was 55.91% at the 5^{th} mode. However, as mentioned previously, such variation is not high, since damping is the most difficulty dynamic parameter to be obtained.

5. ACKNOWLEDGEMENTS

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