A SMOOTHING TECHNIQUE FOR DISCONTINUOUS BOUNDARY ELEMENTS RESULTS IN 2D ELASTICITY

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Abstract. *The use of discontinuous elements in the boundary element method (BEM) does not provide continuous results across de boundary mesh, i.e. variables are not single valued across element interfaces. The implementation of a smoothing technique, able to retrieve continuous results for isoparametric discontinuous boundary elements in two-dimensional elasticity is proposed. The methodology is based on the recovery of smoothed values at the geometric nodes shared by two elements, using least squares fit of the physical nodes values in the neighborhood. New solutions with the same degree of interpolation of the original ones are obtained in each element from the recovered values and, consequently, a continuous solution can be achieved. Moreover, continuous as well as discontinuous boundary elements generate discontinuous, low accuracy results for the tangential component of stress, which is usually obtained by post-processing. This paper presents a new proposal for computing that stress component in discontinuous elements. The proposed technique is based on the use of a smaller number of points with higher convergence in the application of the Hooke's law. The efficiency of the proposed techniques are verified by solving static elasticity problems using linear and quadratic elements.*

Keywords*: boundary elements method, discontinuous elements, variable recovery, variable smoothing, tangential stress*

1. INTRODUCTION

The relaxation of continuity conditions in discretization-based methods has gained impulse among several numerical methods in the last decade. The use of discontinuous elements in boundary element methods (BEM) is somewhat old, but discontinuous Galerkin finite element methods (FEM) are prime examples of the developments in this field. These approaches greatly simplify the computational implementation of the solution methods, and may increase their efficiency, particularly in nonlinear problems or problems containing discontinuous fields.

In the BEM context, there are a number of advantages in the use of discontinuous boundary elements in spite of the characteristic interelement discontinuities. Discontinuous interpolation presents C^1 continuity on all physical nodes, which simplifies the computation of strongly singular integrals. It also avoids the need of double nodes in cases containing corners and discontinuities in the boundary conditions. In addition, the use of discontinuous elements has already proved its efficiency in the solution of multi-domain BEM formulations and FEM-BEM couplings (Zhang and Zhang, 2002). On the other hand, the recovery of variables at the ends of the elements by simple extrapolation or by averaging the extrapolated results of two or more elements are usually inadequate.

A similar problem occurs in stress evaluation by FEM, since the optimal values must be computed at non-nodal points, resulting a discontinuous – element by element – stress field which must be smoothed in order to recover nodal values. This is generally done as a post-processing step, and among the several techniques developed one can mention extrapolation from the Barlow points (Barlow, 1976), global and local L_2 smoothing (Hinton and Campbell, 1974) and the various types of superconvergent patch recovery (SPR) procedures (Zienkiewicz and Zhu, 1992), the latter being possibly the most used.

The present work presents the application of one-dimensional SPR methods to post-process results obtained by discontinuous boundary elements. Although only elasticity problems are illustrated, the basic procedure is valid for any governing equation. It is shown that continuous and better defined interelement results are obtainable, thus eliminating one of the major drawbacks in discontinuous BEM formulations.

Another issue approached by the present work is related to the boundary stress components not evaluated directly from the BEM boundary solution. It is well known that in 2D elasticity problems only of two stress components are directly given by the traction components along the boundary. The remaining component of the stress tensor must be computed by mixing the known stress components and another term, evaluated by differentiating the shape functions in order to estimate the normal strain in the tangential direction. Regardless the elements are continuous or not, lower accuracy is generally found for these post-processed stress components due to the reduction by one degree in the approximation polynomial. Therefore, one can expect problems similar to those found in FEM for Mindlin plates, where the shear strain is evaluated by mixing p polynomials for the plate rotations with p-1 polynomials for the derivatives of the transverse displacement. This is not a robust method because of two reasons: (a) mixing primal variables with dual ones (obtained by numerical differentiation) may lead to ill-conditioned equations (Guiggiani, 1994); (b) nodes are not the optimal ordinates to recover derivative (dual) variables. Although it is a viable technique for many applications, the tangential stress component may present significant errors when coarse meshes are employed. Aiming the evaluation of more reliable values for the stress components on the boundary, a low-cost alternative technique for computing the normal tangential stress component is presented and tested in this work.

Both the proposed smoothing and the alternative technique for evaluation of the tangential stress are implemented for linear and quadratic discontinuous boundary elements, and used to solve 2-D elasticity benchmarks. The results obtained are compared with the conventional BEM results.

2. USING SPR METHODS IN DISCONTINUOUS BOUNDARY ELEMENTS

This section presents a smoothing procedure very similar to the SPR methods used in FEM, but devoted to recovering the results on the geometric nodes of discontinuous boundary elements. The underlying objective is to avoid direct variable extrapolation from the physical nodes to the geometric (end) nodes through element shape functions. This not only leads to inaccurate results at element extremities (particularly when the element offset is large), but also will require further weightening of the results in order to obtain the smoothed value at the shared node.

Let the domain boundary be defined by line segments, which are divided into discontinuous boundary elements. The objective of the variable recovery technique is to find a continuous field (stress or displacements) along the boundary segments using a set of recovered nodal parameters $\underline{u}^*$:

$$u^* = \mathbf{N}\underline{u}^* \tag{1}$$

where **N** are the geometric interpolation functions of the elements discretizing the current segment.

In the present implementation, a first order neighborhood was used, i.e., the two elements will form the patch containing the shared geometric node. Figure 1 illustrates the patches for linear and quadratic discontinuous elements.

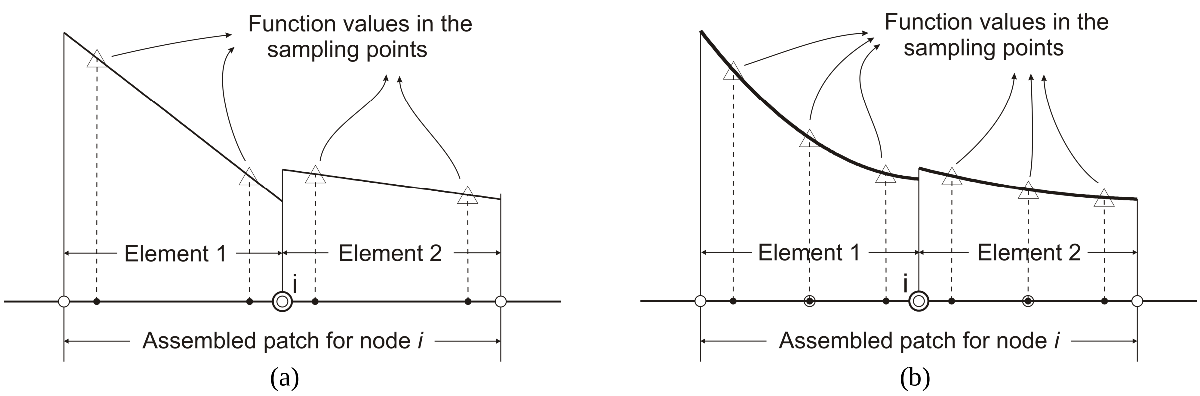

Figure 1. Discontinuous boundary element patches. (a) linear elements; (b) quadratic elements.

It is assumed that the recovered nodal values $\underline{u}^*$ belong to a polynomial expansion u_p^*, which is valid over an element patch surrounding the geometric node considered. This polynomial expansion is one degree higher than the base functions **N** and can be written as:

$$u_p^* = \mathbf{Pa} \tag{2}$$

where u_p^* is any stress or displacement component (σ_1, σ_2, σ_{12}, u_1, u_2), **P** contains the appropriate terms of a complete polynomial of order p, and **a** contains generalized parameters to be determined.

The evaluation of the unknown parameters **a** of the expansion in Eq. (2) is accomplished by a least square fit of u_p^* using the element results at the sampling points (physical nodes) along the patch considered. Therefore, one has to find **a** which minimizes the function:

$$F(\mathbf{a}) = \sum_{i=1}^{m}\left(u_h(x_i) - u_p^*(x_i)\right)^2 = \sum_{i=1}^{m}\left(u_h(x_i) - \mathbf{P}(x_i)\mathbf{a}\right)^2 \tag{3}$$

where x_i are the local coordinates of the sampling points and m is the total number of sampling points in the patch. Here, m = 4 for linear elements (two physical nodes in each element × two elements in the patch), while m = 6 for quadratic elements (three physical nodes in each element × two elements in the patch). The minimization of $F(\mathbf{a})$ leads to an algebraic system that can be solved as

$$\mathbf{a} = \mathbf{A}^{-1}\mathbf{b} \tag{4}$$

where

$$\mathbf{A} = \sum_{i=1}^{m} \mathbf{P}^{\mathrm{T}}(x_i)\mathbf{P}(x_i) \qquad \text{and} \qquad \mathbf{b} = \sum_{i=1}^{m} \mathbf{P}^{\mathrm{T}}(x_i) u_h(x_i) \tag{5}$$

Once the parameters **a** are determined the recovered value is computed at any position of the patch by inserting the appropriate coordinates in Eq (2). The recovery is made only for the shared node of each path. Thus, the value for the central node in each quadratic element is calculated by the average value obtained by the two closest patches. The values in the extreme of a segment of contour are calculated by the nearest patch. Figure 2 shows graphically the variable recovery for linear and quadratic elements. Similar ideas can be used in higher order elements.

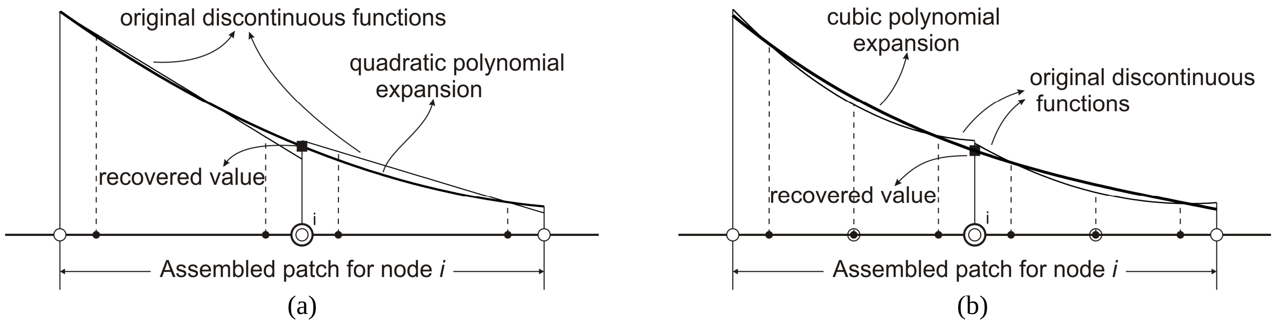

Figure 2. Variable recovery on a shared node. (a) Linear elements; (b) Quadratic elements.

3. AN ALTERNATIVE TECHNIQUE FOR TANGENTIAL STRESS CALCULATION IN DISCONTINUOUS BOUNDARY ELEMENTS

In numerical analysis, the computation of quantities by combining interpolated values and its derivatives must be done with care, as the optimal sampling points of the derivatives are not coincident with the interpolation points themselves. This issue is relatively common in many branches of computational mechanics. Prime examples can be found in FEM, for instance, in the calculation of stress in two-dimensional elasticity elements, in the evaluation of shear strains in structural elements (plate/shell), or in the pressure-velocity coupling in fluid mechanics. This is essentially the very same problem that causes the locking phenomenon of in low order thick plate finite elements (Oñate et al. 1992, Zienkiewicz et al. 1993). A similar problem occurs in the standard evaluation of the tangential boundary stress components for elasticity in BEM, although not characterized by the same consequences as in FEM.

As aforementioned, the missing boundary stress components in the conventional BEM are obtained combining the boundary tractions with the tangential strain is obtained by differentiation of interpolation functions over each element (Brebbia et al. 1984). It is known that this technique not necessarily provides good results along the whole element (Guiggiani, 1994). Here, it is suggested a small change in the use of Hooke's law in order to obtain a more reliable estimate of the tangential stress component for boundary elements without any significant increase in the computational cost. Basically, the tangential strain is sampled at optimal locations, instead at the nodes. The ideas presented herein are implemented and tested for 2D elasticity discontinuous boundary elements, but they can be used in a fairly broad class of problems, regardless the continuity of the interpolation.

3.1. Standard technique for tangential stress calculation

In 2D elasticity problems, the normal (σ_{nn}) and shear (σ_{nt}) boundary stress are directly related to the boundary tractions (p_n, p_t) in a local coordinate system (n, t). Assuming that the tractions are written in the global coordinate system, the boundary stress components are easily obtained by rotating the tractions according to the local system:

$$\begin{Bmatrix} \sigma_{nn} \\ \sigma_{nt} \end{Bmatrix} = \mathbf{R} \begin{Bmatrix} p_1 \\ p_2 \end{Bmatrix} \tag{6}$$

where α is the angle between global and the local coordinate systems (Fig. 3). The tangential strain (ε_{tt}) is obtained by using the interpolated displacements (Zhao, 1996):

$$\varepsilon_{tt} = \frac{du_t}{dt} = \frac{du_1}{dt}t_1 + \frac{du_2}{dt}t_2 \qquad \therefore \qquad \varepsilon_{tt} = \frac{1}{J}\sum\left(\frac{d\phi_i(\xi)}{d\xi}u_1^i t_1 + \frac{d\phi_i(\xi)}{d\xi}u_2^i t_2\right) \tag{7}$$

where t_1 and t_2 are components of the unit tangential vector in x_1 and x_2 directions, respectively, u_1^i and u_2^i are the nodal displacements in the i-th node in x_1 and x_2 directions, respectively; and $\phi_i(\xi)$ are the physical interpolation functions (J is the Jacobian of the element transformation to the normalized space).

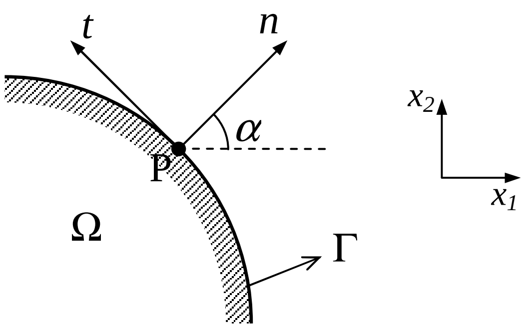

Figure 3. Coordinate system over the boundary.

The tangential stress component (σ_{tt}), can be obtained by Hooke's law for plane-strain:

$$\sigma_{tt}(\xi) = \frac{1}{1-\nu}\left[\nu\,\sigma_{nn}(\xi) + 2G\varepsilon_{tt}(\xi)\right] \tag{8}$$

where ν is the Poisson's ratio, and G is the shear modulus.

Therefore, when Eq.(8) is used in the standard BEM, it sums two polynomial terms of different orders. Depending on where σ_{tt} is evaluated, this procedure process may lead to unreliable results unless the ξ co-ordinate is known to be an optimal point to retrieve derivative quantities (present in the ε_{tt} term).

3.2. Alternative technique for tangential stress calculation

The existence of points able to represent optimally the derivative of an interpolated function is well known and can be proved mathematically. In the FEM, these points are known as Barlow points, and they are used to evaluate stress fields from differentiation of interpolated displacements (Barlow, 1976 and Prathap, 1996). When the interpolation function is of the polynomial type, these points are located at the Gauss-Legendre stations corresponding to one order less than the minimum order necessary to integrate the interpolation function exactly. The underlying idea of the scheme proposed here is to use these points to evaluate Eq.(8). To the best of the authors' knowledge, there are no similar studies correlating these aspects in the BEM context.

In the case of linear boundary elements, the normal stress is obtained directly from the traction forces, and therefore it is a linear function (as well as the displacements). The tangential deformation is represented by a constant function in each element since it is obtained by the displacement derivative. The combined use of these two functions, through the Hooke's law, is the origin of often unsatisfactory results (Guiggiani, 1994). The present work suggests the use of the central point of the element – Gauss point for a linear function integration ($\xi = 0$), to sample the differentiation of the interpolated displacement. The co-ordinate $\xi = 0$ delivers the best estimate for the tangential strain along the element. It is worth to note that the evaluation of this strain at nodal locations ($\xi = \pm 1$) will overestimates or underestimates the strain value. In summary, it is proposed that both, the normal stress and tangential strain should be evaluated at the center of the element, thus obtaining a constant function for tangential stress over each element. For clarity, Table 1 compares both ways for the evaluation of the tangential stress in linear elements.

For quadratic discontinuous elements where, a priori, the normal stress and tangential strain are represented by quadratic and linear functions respectively, it is suggested that the Gauss points for a cubic quadrature ($\xi = \pm 1/\sqrt{3}$) should be used to represent the tangential strain field along the element. Therefore, replacing the standard technique, the calculation of the tangential stress is performed using the values for normal stress and tangential strain just at two points. A linear interpolation of the values obtained in the Gauss points is made in order to obtain the nodal stress values of the quadratic element. Table 2 shows the two methods for tangential stress calculation on quadratic elements.

In summary, the tangential stress calculation is made with one degree less than the other stress components. This method may initially seem less sound than the conventional procedure, but later it will be shown that when used with the smoothing technique described in section 2, the proposed scheme leads to better results. In many cases, it was found that the conventional scheme will produce wrong signs to the ε_{tt} term in Eq.(8), a direct consequence of it being sampled at non-optimal points.

Table 1. Tangential stress calculation for linear discontinuous boundary elements*.

STANDARD TECHNIQUE	ALTERNATIVE TECHNIQUE
$\sigma_{tt}^1 = \frac{1}{1-\nu}\left(\nu\sigma_{nn}^1 + 2G\varepsilon_{tt}^1\right)$ $\sigma_{tt}^2 = \frac{1}{1-\nu}\left(\nu\sigma_{nn}^2 + 2G\varepsilon_{tt}^2\right)$	$\sigma_{tt}(\xi_1) = \frac{1}{1-\nu}\left(\nu\sigma_{nn}(\xi_1) + 2G\varepsilon_{tt}(\xi_1)\right)$ $\sigma_{tt}^1 = \sigma_{tt}^2$ refers to constant interpolation of $\sigma_{tt}(\xi_1)$

* $\xi_1 = 0$, and the superscript represent the associated nodal value of the variable.

Table 2. Tangential stress calculation for quadratic discontinuous boundary elements*.

STANDARD TECHNIQUE	ALTERNATIVE TECHNIQUE
$\sigma_{tt}^1 = \frac{1}{1-\nu}\left(\nu\sigma_{nn}^1 + 2G\varepsilon_{tt}^1\right)$ $\sigma_{tt}^2 = \frac{1}{1-\nu}\left(\nu\sigma_{nn}^2 + 2G\varepsilon_{tt}^2\right)$ $\sigma_{tt}^3 = \frac{1}{1-\nu}\left(\nu\sigma_{nn}^3 + 2G\varepsilon_{tt}^3\right)$	$\sigma_{tt}(\xi_1) = \frac{1}{1-\nu}\left(\nu\sigma_{nn}(\xi_1) + 2G\varepsilon_{tt}(\xi_1)\right)$ $\sigma_{tt}(\xi_2) = \frac{1}{1-\nu}\left(\nu\sigma_{nn}(\xi_2) + 2G\varepsilon_{tt}(\xi_2)\right)$ σ_{tt}^1 , σ_{tt}^2 e σ_{tt}^3 are obtained by linear interpolation of $\sigma_{tt}(\xi_1)$ e $\sigma_{tt}(\xi_2)$

* $\xi_1 = -1/\sqrt{3}$, $\xi_2 = 1/\sqrt{3}$, and the superscript represent the associated nodal value of the variable.

4. NUMERICAL RESULTS

In order to investigate the performance of the smoothing procedure and the alternative tangential stress evaluation scheme, this section shows some results obtained using both. Numerical integration was carried out using 16 Gauss points, in order to minimize the influence of quadrature errors. Dimensions, material properties, and other physical data are given without units, but they were specified to represent a compatible system of units. The material properties used in all case are: E = 210e9 and ν = 0.3 Plane-stress condition is assumed throughout this section.

4.1. Square-plate with a central hole under traction

A 100×100 square plate with a central hole of radius R = 5 was analyzed. Due to symmetry, only one quarter of the plate was considered (Fig. 4). The traction loading along the upper side was set to P = 1. The offset of all boundary elements used in the mesh is 15% of the element length.

Linear and quadratic elements were used with two different meshes for each type of element. Mesh 1 used an element size of 2.5 along the straight boundaries and four elements along the quarter-circle. Mesh 2 used an element size of 1.25 and eight elements along the quarter-circle.

Figures 5 and 6 exemplify the benefits of the smoothing procedure by plotting the hoop stress along the edge AB. The analytical solution for an infinite plate with central hole is compared against the raw and smoothed results obtained by BEM. One can see clearly in Figs. 5-6 that the smoothing do not destroy the overall behavior of the variable, and more importantly, it preserves peak values like the stress concentration on the hole (point A). It is also worth to note that the recovered values along the element interfaces differ considerably from the value obtained by extrapolation of the discontinuous results.

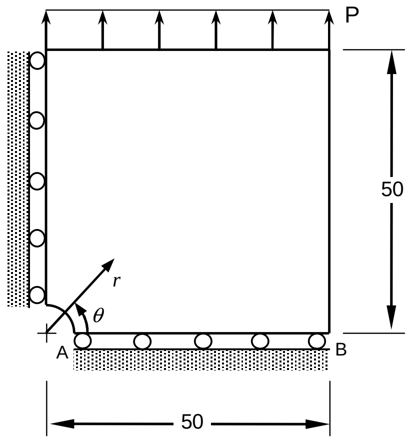

Figure 4. Squared plate with a central hole under uniform traction.

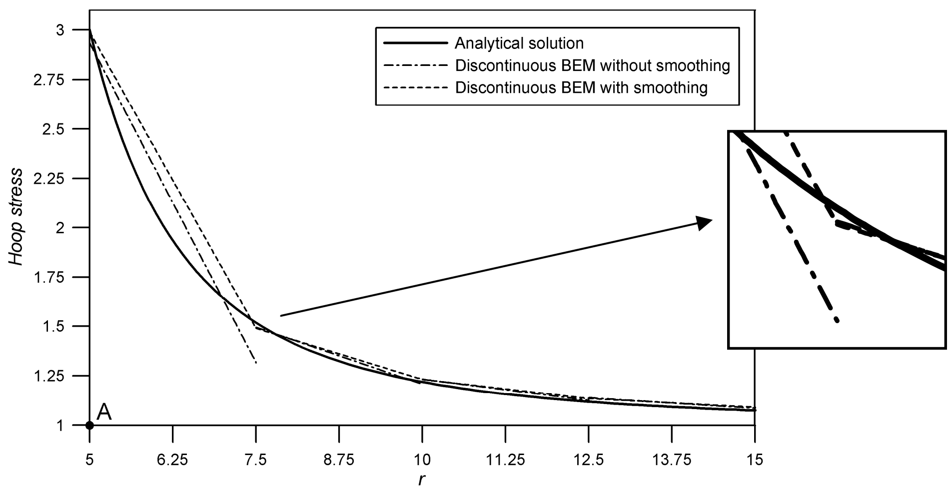

Figure 5. Hoop stress smoothing results for squared plate with central hole. Linear elements – mesh 1.

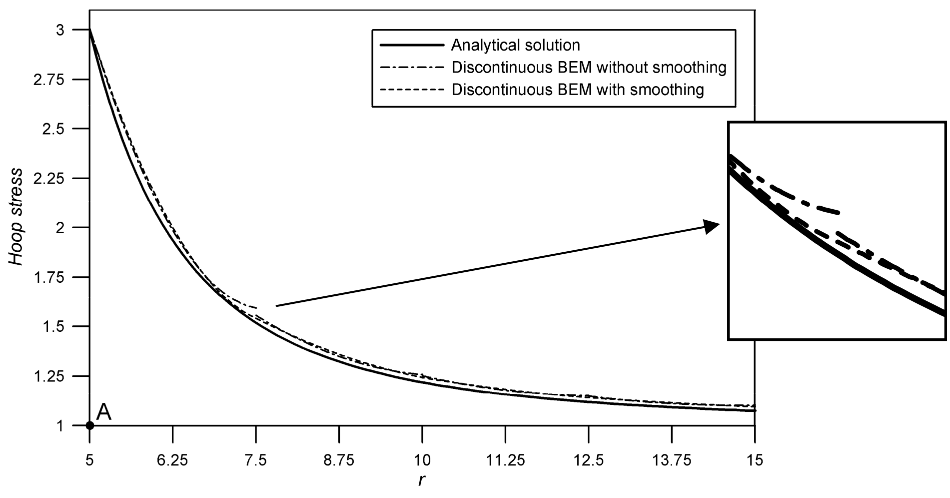

Figure 6. Hoop stress smoothing results for squared plate with central hole. Quadratic elements – mesh 1.

Regarding the application of the alternative method for tangential stress calculation, it can be used with or without the smoothing procedure, leading to four possibilities for post-processing the results:

- Method A: Discontinuous BEM without smoothing – the raw results of discontinuous elements are considered with standard tangential stress calculation (section 3.1).
- Method B: Discontinuous BEM with smoothing – same as Method A, but the results are smoothed.
- Method C: Modified discontinuous BEM without smoothing – raw results of discontinuous elements with alternative tangential stress calculation as outlined in section 3.2.
- Method D: Modified discontinuous BEM with smoothing – same as Method C, but the results are smoothed.

These methods were used to post-process the normal radial stress along the edge AB, which is the tangential component along that piece of boundary. Figures 7 and 8 compare graphically these results for linear elements with meshes 1 and 2.

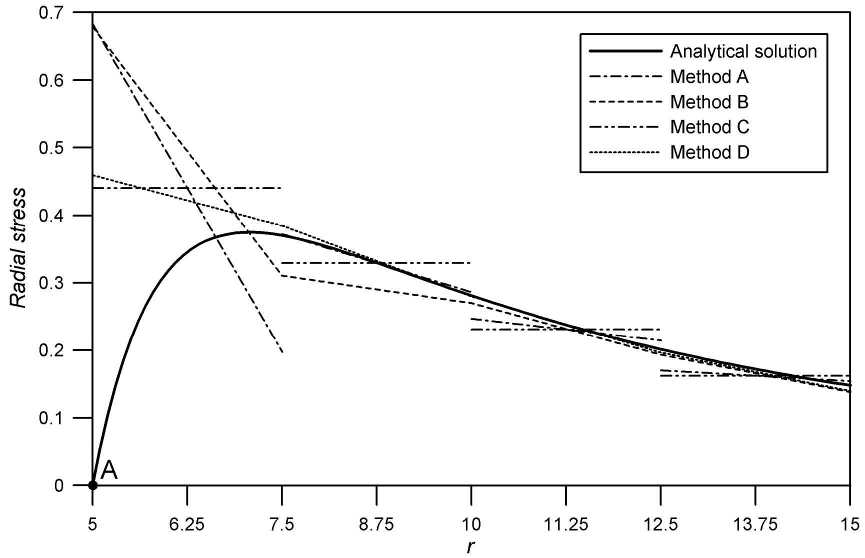

Figure 7. Radial stress recovery along edge AB. Linear elements – mesh 1.

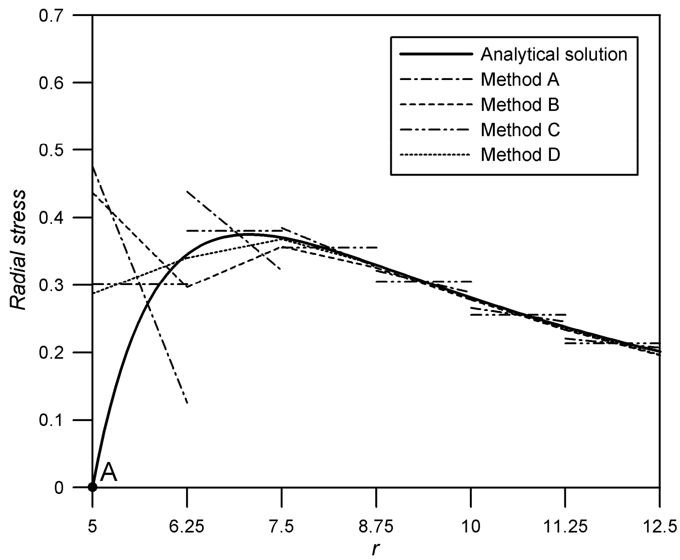

Figure 8. Radial stress recovery along edge AB. Linear elements – mesh 2.

The graphs depicted in Figs.7-8 show that the response which agrees more closely to the analytical solution is the smoothed solution considering the alternative tangential stress calculation. As expected, it can be seen that the alternative tangential solution without smoothing is simply an element average value from discontinuous BEM without smoothing.

Figure 9 shows the recovery of radial stress on the same edge, this time using quadratic elements. As in Figs.7-8, these graphs show the four types of post-processed results against the analytical solution. Although the differences between the four methods are not as drastic as in the case of linear elements, it is evident that the smoothed results obtained with alternative tangential stress calculation agree more closely to the analytical solution, particularly at point A. Another important aspect is that the differences between all methods tend to vanish where analytical solution is less oscillatory (away from stress concentration areas).

Interestingly, it is also evident from Figs.7-9 that none of the methods provided very good results near the hole, although the modified stress calculation seems to recover the better ones. This is direct consequence of the different signals of the terms in Eq.8, i.e. the high gradients of the tangential strain near the hole are miscalculated when the displacements are differentiated at non-optimal locations. Of course, this effect becomes more conspicuous when coarse meshes are used.

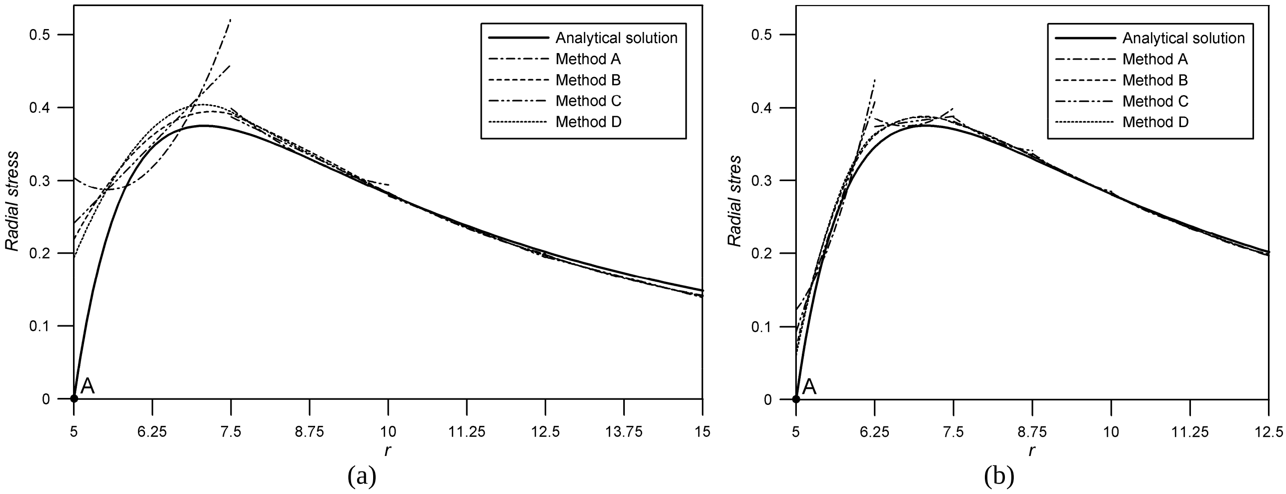

Figure 9. Radial stress recovery. Quadratic elements: (a) mesh 1, (b) mesh 2.

4.2. L-shaped domain

Figure 10 shows the geometry and boundary conditions of an L-shaped plate. This problem is a typical test for adaptive meshing procedures in both the FEM and the BEM (Gago et al., 1983, Zhao e Wang, 1999, Zienkiewicz e Zhu, 1992b). It was also used as a numerical example to demonstrate the accuracy of a formulation for interelement stress evaluation by boundary elements (Zhao, 1996).

Due to a re-entrant corner the normal stress in the radial direction have infinite value at point B. According to Guiggiani (1990), the displacement field near the internal corner behaves as Kr^{α}, where K is the stress concentration factor, r the distance from the corner and α the strength of the singularity. By contrast, all boundary unknowns are still bounded and there is only a singularity in the contour derivative of the boundary displacement at corner B. Moreover, because there are no tractions on the internal edges of the ABC corner, the radial stress (tangential stress in the local coordinates) is obtained through the Hooke's law from the tangential strain (differentiation of displacements), only. Thus, this stress component has one degree less in its interpolation than the element interpolation. For all that, the two techniques described for tangential stress calculation give identical results on the edge examined in this example.

The edge BC of the internal corner was used to verify the smoothing scheme. Two meshes were employed: mesh 1, totaling 32 elements, and mesh 2 with 64 elements. 4 and 8 elements were used along edge BC, for each mesh, respectively. Additionally, a third, finer mesh (mesh 3) with a total of 268 elements, being 54 of them on edge BC, was used as a reference value. The three cases were analyzed with quadratic elements with P = 1). The offset used in all elements was 10% of the element lengths. Figure 11 shows the radial stress results obtained for meshes 1 and 2 up to half of the edge BC. The smoothing scheme generated satisfactory results from the half of the first element, with errors of less than 1.5%, as shown in Fig. 11a. However, the errors are considerably larger along the area closer to point B. Moreover, the raw solution is linear, while the smoothed solution is quadratic (same degree of interpolation of the element).

In order to assess the influence of the mesh discretization, Fig. 11b shows the results for this case using a mesh with twice the number of elements (mesh 2). The result obtained show good agreement to the smoothed solution of mesh 3, except for $r < 0.45$. Errors related to the radial stress at r = 2.5 (position share by two elements in all meshes), using different methodologies to calculate the interelement stress, are shown in Table 3. The relative error is calculated as:

$$Error = \left| \frac{\sigma_{mesh1,2} - \sigma_{mesh3}}{\sigma_{mesh3}} \right| \times 100 \quad (9)$$

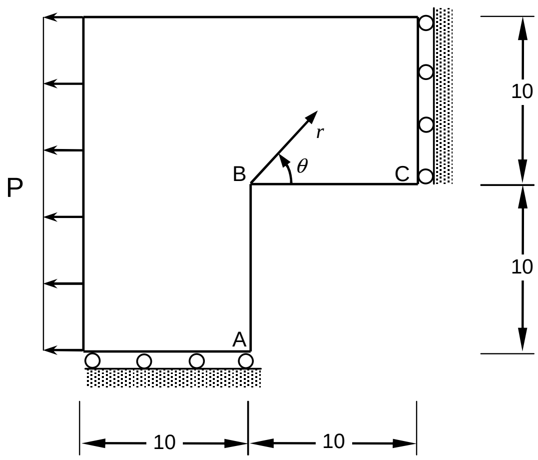

Figure 10. L-shaped domain.

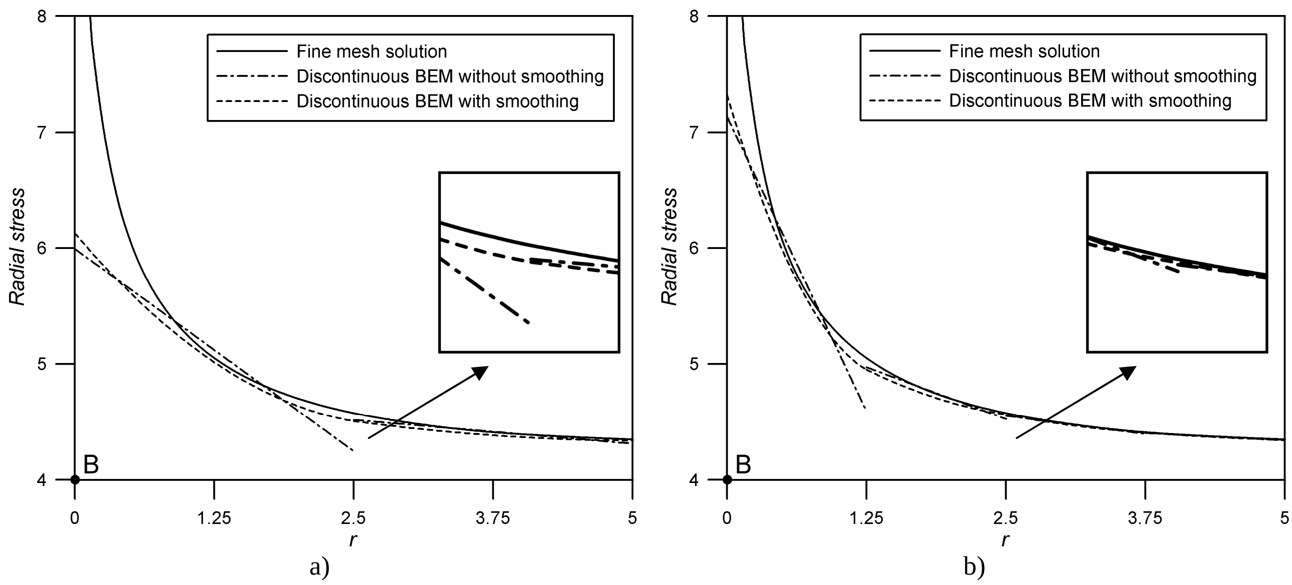

Figure 11. Radial stress recovery along edge BC. Quadratic elements: (a) mesh 1, (b) mesh 2.

Table 3. Radial stress results at $r = 2.5$ (L-shaped domain)

MESH	METHOD	ERROR
Mesh 1	left extrapolation	7.149 %
	right extrapolation	1.268 %
	left and right extrapolation average	4.198 %
	smoothed solution	1.487 %
Mesh 2	left extrapolation	1.093 %
	right extrapolation	0.481 %
	left and right extrapolation average	0.787 %
	smoothed solution	0.284 %

From the results of Tab.3 it is clear that the errors of the smoothed solution are smaller than any type of direct extrapolation. However, it is interesting to note that, since the first element along each boundary segment presents a linear interpolation, there may be some loss of quality in the results when very coarse meshes are used.

5. CONCLUSIONS

It was developed a method to obtain smoothed results for discontinuous boundary elements in two-dimensional elasticity, based on SPR methods used in FEM context. More accurate interelement values were obtained in comparison to direct extrapolation of the original discontinuous solutions. The proposed technique performs efficiently to recover results in both types of discontinuous elements. Different meshes used indicated that the smoothed solution becomes more accurate when the mesh is refined. Moreover, the smoothed solutions converge faster to the reference solutions.

Another goal of the present work was the development of an alternative technique for tangential stress component calculation, which showed to estimate more reliable results when compared to the standard boundary stress technique.

Dependence on other factors should be studied, such as number of integration points, offset values, and meshes with variable element sizes.

Finally, the increase in the total computational cost is negligible when compared to other processing stages. Both schemes proposed present potential to be used with other types of discontinuous elements or different governing equations without hurdles.

6. ACKNOWLEDGEMENTS

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