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A NUMERICAL METHOD FOR SOLVING THREE-DIMENSIONAL INCOMPRESSIBLE FLUID FLOWS FOR HYDROELECTRIC RESERVOIR APPLICATIONS

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Abstract. *A numerical model is proposed for the Navier-Stokes three-dimensional equations write under Eulerian formulation. The semi-Lagrangian method was used to discretized the convective terms. The linear system is decomposed in LU blocks through the discrete projection method and solved by an iterative method. The three-dimensional domain was represented by a mesh, represented by a topological data structure, formed with cells forming linear wedge elements. Experiments solving a particular problem were made in order to analyze the consistency of the proposed method on theoretical basis. The results showed a good approximation and pointed out the stability of the proposed method.*

Keywords: *Navier-Stokes Equations, Finite Elements Method, Numerical Simulation, Semi-Lagrangean Method*

1. INTRODUCTION

There are many cases in engineering where the simulation of fluid flow becomes necessary. The tridimensional modeling is the most direct form and the one that the most resemble the reality. However, a simple and efficient approach is often desirable avoiding the computational high costs of a 3D simulation.

In the study of incompressible fluid flows, the mathematical modeling of the conservation laws is well stated by Navier–Stokes equations and the mass conservation equations. The need for numerical simulation in CFD is justified by the lack of analytical solutions of the Navier-Stokes equations for most practical cases.

In this paper, a numerical model is proposed for the solution of three dimensional Navier–Stokes equations (momentum and mass conservation equations). The finite elements method (Zienkiewicz, 2000)(Becker et al., 1981)(Zienkiewicz and Cheung, 1965)(Chung, 1978) is used for the discretization of the proposed problem, where the Galerkin method is used for the spatial discretization and the semi-lagrangean method is used for the discretization of the material derivative. The latter derivative includes the convective term, responsible for the non linearity of the problem. The experiments showed properties of a very needed hydroelectric model simulation, where the fluid flows over steps in the reservoir.

2. FORMULATION

The governing equations are the non-dimensional mass and momentum equations in conservative form where in three-dimensional coordinates can be written as

$$\frac{D(\rho\mathbf{u})}{Dt} = -\nabla p + \frac{1}{Re}\nabla\cdot[\mu(\nabla\mathbf{u} + \nabla\mathbf{u}^T)] + \frac{1}{Fr^2}\rho\mathbf{g} \tag{1}$$

and the equation of continuity

$$\nabla\cdot\mathbf{u} = 0. \tag{2}$$

where $Re = (LU)/\nu$ and $Fr = U/(\sqrt{gL})$ are the non-dimensional Reynolds and Froude numbers, respectively. Hence, L and U are the length and velocity scales, respectively, ν is the kinematic viscosity, and g denotes the gravitational constant, $g = |\mathbf{g}| = |(g_x, g_y, g_z)|$. Furthermore, $\mathbf{u} = (u, v, w)^t$ is the velocity vector while p is the non-dimensional pressure.

2.1 Variational Formulation

Considering the Navier–Stokes equations for incompressible flows, written in the eulerian formulation expressed in the non-dimensional form as

$$\frac{D(\rho\mathbf{u})}{Dt} + \nabla p - \frac{1}{Re}\nabla\cdot\left[\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^{\mathbf{T}}\right)\right] - \frac{1}{Fr^2}\rho\mathbf{g} = 0 \tag{3}$$

$$\nabla\cdot\mathbf{u} = 0 \tag{4}$$

valid on a domain $\Omega \subset \mathbb{R}^m$ under the boundary conditions

$$\mathbf{u} = \mathbf{u}_\Gamma, \text{ em } \Gamma_1 \tag{5}$$

$$u_t = 0 \text{ e } \sigma^{\text{nn}} = 0, \text{ em } \Gamma_2. \tag{6}$$

Consider the subspace

$$\mathbb{V} = H^1(\Omega)^m = \left\{\mathbf{v} = (v_1,\ldots,v_m) : v_i \in H^1(\Omega), \forall i = 1,\ldots,m\right\} \tag{7}$$

where $H^1(\Omega)$ is the *Sobolev* space given by

$$H^1(\Omega) = \left\{v \in L^2(\Omega) : \frac{\partial v}{\partial x_i} \in L^2(\Omega), i = 1,\ldots,m\right\} \tag{8}$$

with $L^2(\Omega)$ being a infinity dimension space defined as

$$L^2(\Omega) = \left\{v : \Omega \to \mathbb{R}, \int_\Omega v^2 d\Omega < \infty\right\} \tag{9}$$

And $\mathbb{V} = H^1(\Omega)^m$ is the cartesian product of m spaces $H^1(\Omega)$.

Defining

$$\mathbb{V}_{\mathbf{u}\Gamma} = \mathbf{v} \in \mathbb{V} : \mathbf{v} = \mathbf{u}_\Gamma \text{ in } \Gamma_1,\ \mathbb{V}_0 = \mathbf{v} \in \mathbb{V} : \mathbf{v} = \mathbf{0} \text{ in } \Gamma_1 \tag{10}$$

$$\mathbb{P}_{p\Gamma} = q \in L^2(\Omega) : q = p_\Gamma \text{ in } \Gamma_2 \tag{11}$$

the weak formulation of the problem can be written as: find $\mathbf{u}(\mathbf{x},t) \in \mathbb{V}_{\mathbf{u}\Gamma}$ e $p(\mathbf{x},t) \in \mathbb{P}_{p\Gamma}$ such that

$$\int_\Omega \frac{D(\rho\mathbf{u})}{Dt}\cdot\mathbf{w}d\Omega - \int_\Omega \nabla p\cdot\mathbf{w}d\Omega - \int_\Omega \frac{1}{Re}\nabla\cdot\left[\mu\left(\nabla\mathbf{u}+\nabla\mathbf{u}^T\right)\right] : \mathbf{w}d\Omega - \int_\Omega \frac{1}{Fr^2}\rho\mathbf{g}\cdot\mathbf{w}d\Omega = 0.$$

$$\int_\Omega (\nabla\cdot\mathbf{u})\, q d\Omega = 0 \tag{12}$$

for all $\mathbf{w} \in \mathbb{V}_0$ e $q \in \mathbb{P}_{p\Gamma}$.

The discretization of (12) is made by using linear shape functions and Galerkin weighting functions. Integrating over the wedge elements results in an ODE system, which is solved using the projection method described as follows. The time derivatives are integrated by an implicit scheme. As in the Lagrangian formulation the non-linear terms do not appear, the matrices are defined symmetric positive and thus the conjugate gradient method can be applied to solve the linear systems.

2.2 Galerkin's Method

After the variational formulation of the governing equations, the approximation phase takes place using the Galerkin's method. Consider the governing equations in its non-dimensional and variational form (Eq. 12) and letting NV be the number of velocity points, NP the number of pressure points and NE the number of finite elements of the mesh that discretizes the domain Ω. The Galerkin's method consists on replacing the following terms on Eq. (12):

$$u(\mathbf{x},t) \approx \sum_{n=1}^{NV}\psi_n(\mathbf{x})u_n(t), \qquad v(\mathbf{x},t) \approx \sum_{n=1}^{NV}\psi_n(\mathbf{x})v_n(t) \tag{13}$$

$$w(\mathbf{x},t) \approx \sum_{n=1}^{NV} \psi_n(\mathbf{x}) w_n(t), \qquad p(\mathbf{x},t) \approx \sum_{n=1}^{NP} P_n(\mathbf{x}) p_n(t) \tag{14}$$

that are semi-continuous approximations, that is, continuous in time (t) and discrete in space (x). Here, $\psi_n(x)$ represent the interpolation functions used for the velocity and $P_n(x)$ the interpolating functions for the pressure.

The momentum equation is normally evaluated in all the free nodes of velocity, and then the weight functions w_x, w_y and w_z are replaced by interpolation functions $\psi_m = \psi_m(x)$, $m = 1, ..., NV$. Applying this procedure for the directions x, y and z, and restricting the nodal interpolation functions to each element e, in the direction x, we have

$$\sum_e \int_{\Omega_e} \sum_{i,j \in e} \rho^e \frac{Du_j}{Dt} \psi_i^e \psi_j^e d\Omega - \sum_e \int_{\Omega_e} \sum_{i,k \in e} \frac{\partial \psi_i^e}{\partial x} P_k^e p_k d\Omega - \frac{1}{Re} \sum_e \int_{\Omega_e} \sum_{i,j \in e} \mu^e \left(\frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} u_j + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} u_j + \right.$$
$$\left. \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial z} u_j + \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} u_j + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial x} v_j + \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial x} w_j \right) d\Omega - \frac{1}{Fr^2} \sum_e \int_{\Omega_e} \sum_{i,j \in e} \rho^e \psi_i^e \psi_j^e g_{x,j} d\Omega = 0 \tag{15}$$

In the direction y,

$$\sum_e \int_{\Omega_e} \sum_{i,j \in e} \rho^e \frac{Dv_j}{Dt} \psi_i^e \psi_j^e d\Omega - \sum_e \int_{\Omega_e} \sum_{i,k \in e} \frac{\partial \psi_i^e}{\partial y} P_k^e p_k d\Omega - \frac{1}{Re} \sum_e \int_{\Omega_e} \sum_{i,j \in e} \mu^e \left(\frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} v_j + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} v_j + \right.$$
$$\left. \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial z} v_j + \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial y} u_j + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} v_j + \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial y} w_j \right) d\Omega - \frac{1}{Fr^2} \sum_e \int_{\Omega_e} \sum_{i,j \in e} \rho^e \psi_i^e \psi_j^e g_{y,j} d\Omega = 0 \tag{16}$$

In the direction z,

$$\sum_e \int_{\Omega_e} \sum_{i,j \in e} \rho^e \frac{Dw_j}{Dt} \psi_i^e \psi_j^e d\Omega - \sum_e \int_{\Omega_e} \sum_{i,k \in e} \frac{\partial \psi_i^e}{\partial z} P_k^e p_k d\Omega - \frac{1}{Re} \sum_e \int_{\Omega_e} \sum_{i,j \in e} \mu^e \left(\frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial x} w_j + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial y} w_j + \right.$$
$$\left. \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial z} w_j + \frac{\partial \psi_i^e}{\partial x} \frac{\partial \psi_j^e}{\partial z} u_j + \frac{\partial \psi_i^e}{\partial y} \frac{\partial \psi_j^e}{\partial z} v_j + \frac{\partial \psi_i^e}{\partial z} \frac{\partial \psi_j^e}{\partial z} w_j \right) d\Omega - \frac{1}{Fr^2} \sum_e \int_{\Omega_e} \sum_{i,j \in e} \rho^e \psi_i^e \psi_j^e g_{z,j} d\Omega = 0 \tag{17}$$

The equation of continuity Eq. (2) is evaluated on the free nodes of pressure, then weight function q is approximated by the interpolation functions associated with the pressure $P_r(x)$, resulting

$$\sum_e \int_{\Omega^e} \sum_n \left(\frac{\partial \psi_n}{\partial x} u_n + \frac{\partial \psi_n}{\partial y} v_n + \frac{\partial \psi_n}{\partial z} w_n \right) P_r d\Omega = 0 \tag{18}$$

for $r = 1, \dots, NP$. Restricting the interpolation functions to each element e, we have

$$\sum_e \int_{\Omega^e} \sum_{j,k \in e} \left(\frac{\partial \psi_j^e}{\partial x} u_j + \frac{\partial \psi_j^e}{\partial y} v_j + \frac{\partial \psi_j^e}{\partial z} w_j \right) P_k^e d\Omega = 0 \tag{19}$$

The Eq. (15), (16) e (17) can be represented in an ordinary differential equations system form

$$\mathbf{M}_{\rho,x} \frac{Du}{Dt} - \frac{1}{Re} \left((2\mathbf{K}_{xx} + \mathbf{K}_{yy} + \mathbf{K}_{zz})\mathbf{u} + \mathbf{K}_{xy}\mathbf{v} + \mathbf{K}_{xz}\mathbf{w} \right) - \mathbf{G}_x \mathbf{p} - \frac{1}{Fr^2} \mathbf{M}_{\rho,x} \mathbf{g}_x = 0$$

$$\mathbf{M}_{\rho,y} \frac{Dv}{Dt} - \frac{1}{Re} \left(\mathbf{K}_{yx}\mathbf{u} + (\mathbf{K}_{xx} + 2\mathbf{K}_{yy} + \mathbf{K}_{zz})\mathbf{v} + \mathbf{K}_{yz}\mathbf{w} \right) - \mathbf{G}_y \mathbf{p} - \frac{1}{Fr^2} \mathbf{M}_{\rho,y} \mathbf{g}_y = 0$$

$$\mathbf{M}_{\rho,z} \frac{Dw}{Dt} - \frac{1}{Re} \left(\mathbf{K}_{zx}\mathbf{u} + \mathbf{K}_{zy}\mathbf{v} + (\mathbf{K}_{xx} + \mathbf{K}_{yy} + 2\mathbf{K}_{zz})\mathbf{w} + \right) - \mathbf{G}_z \mathbf{p} - \frac{1}{Fr^2} \mathbf{M}_{\rho,z} \mathbf{g}_z = 0$$
$$\mathbf{D}_x \mathbf{u} + \mathbf{D}_y \mathbf{v} + \mathbf{D}_z \mathbf{w} = 0 \tag{20}$$

where $\mathbf{u} = [u_1, \dots, u_{NV}]^T$, $\mathbf{v} = [v_1, \dots, v_{NV}]^T$, $\mathbf{w} = [w_1, \dots, w_{NV}]^T$ $\mathbf{p} = [p_1, \dots, p_{NP}]^T$, $\mathbf{g}_x = [g_1^x, \dots, g_{NV}^x]^T$, $\mathbf{g}_y = [g_1^y, \dots, g_{NV}^y]^T$, $\mathbf{g}_z = [g_1^z, \dots, g_{NV}^z]^T$ are the vectors of the nodal values for the velocity and pressure variables, and the gravity forces.

The dimensions of the matrices of the equations system (20) are $NV \times NP$ for $\mathbf{G}_x$, $\mathbf{G}_y$ and $\mathbf{G}_z$, $NP \times NV$ for $\mathbf{D}_x$, $\mathbf{D}_y$ e $\mathbf{D}_z$ and $NV \times NV$ for all others.

2.3 Semi-Lagrangean Method

This method was introduced in the beginning of the 80's by (Robert, 1981) and (Pironneau, 1982), and the basic idea is based on the discretization of the solution of the Lagrangean derivative in time instead of the eulerian derivative. As an example, one can consider a Semi-Lagrangean scheme of any equation of any type convection-diffusion.

The material derivative of a scalar u is given in the three-dimensional space as

$$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \tag{21}$$

The basic idea of the Semi-Lagrangean method is to follow a fluid particle during its path through the mesh during the flow. The method is explicit, where is necessary the information of the values of the components of velocity in the current time. Therefore, the method approximate those values in the previous step time on the path. Basically, the Semi-Lagrangean formulation is given by

$$\frac{Du}{Dt}(p) = \frac{\mathbf{u}_p^{n+1} - \mathbf{u}_{p^*}^n}{\Delta t} \tag{22}$$

where

$$p^* = p - \Delta t \mathbf{u}_p \tag{23}$$

where p is any point in the mesh and p^* defines the point p in the previous step time. The calculus of $\mathbf{u}$ in the point p^* is made by a linear interpolation between the neighbors points. This interpolations is dependent from where the point p^* is located inside the domain, such as: over an edge, over a vertex, over a face of a wedge element, inside a wedge element or outside the domain.

Then, the equations system (20) can be written as follows.

$$\mathbf{M}_{\rho,x}\left(\frac{u_p^{n+1} - u_{p^*}^n}{\Delta t}\right) - \frac{1}{Re}\left((2\mathbf{K}_{xx} + \mathbf{K}_{yy} + \mathbf{K}_{zz})\mathbf{u} + \mathbf{K}_{xy}\mathbf{v} + \mathbf{K}_{xz}\mathbf{w}\right) - \mathbf{G}_x\mathbf{p} - \frac{1}{Fr^2}\mathbf{M}_{\rho,x}\mathbf{g}_x = 0$$

$$\mathbf{M}_{\rho,y}\left(\frac{v_p^{n+1} - v_{p^*}^n}{\Delta t}\right) - \frac{1}{Re}\left(\mathbf{K}_{yx}\mathbf{u} + (\mathbf{K}_{xx} + 2\mathbf{K}_{yy} + \mathbf{K}_{zz})\mathbf{v} + \mathbf{K}_{yz}\mathbf{w}\right) - \mathbf{G}_y\mathbf{p} - \frac{1}{Fr^2}\mathbf{M}_{\rho,y}\mathbf{g}_y = 0$$

$$\mathbf{M}_{\rho,z}\left(\frac{w_p^{n+1} - w_{p^*}^n}{\Delta t}\right) - \frac{1}{Re}\left(\mathbf{K}_{zx}\mathbf{u} + \mathbf{K}_{zy}\mathbf{v} + (\mathbf{K}_{xx} + \mathbf{K}_{yy} + 2\mathbf{K}_{zz})\mathbf{w} +\right) - \mathbf{G}_z\mathbf{p} - \frac{1}{Fr^2}\mathbf{M}_{\rho,z}\mathbf{g}_z = 0$$

$$\mathbf{D}_x\mathbf{u} + \mathbf{D}_y\mathbf{v} + \mathbf{D}_z\mathbf{w} = 0 \tag{24}$$

3. NUMERICAL METHOD

The numerical procedure implemented to solve the conservation equations is based on the Projection method, initially proposed by (Chorin, 1968), and formalized by (Gresho, 1990)(Gresho and Sani, 1987). Thus, instead of solving one large system, we solve two smaller decoupled systems of equations, reducing the time of computation.

The Projection method based on LU decomposition is obtained though the fatoration in blocks of the resulting linear system. This suggests that the split of the velocity and pressure is made after the discretization in space and in time of the governing equations. Consider the discretized equations in time and space as follows

$$\mathbf{M}\rho\left(\frac{\mathbf{u}^{n+1} - \mathbf{u}_*^n}{\Delta t}\right) - \frac{1}{Re}\mathbf{K}\mathbf{u}^{n+1} - \mathbf{G}\mathbf{p}^{n+1} - \frac{1}{Fr^2}\mathbf{M}_\rho\mathbf{g} = 0 \tag{25}$$

$$\mathbf{D}\mathbf{u}^{n+1} = 0 \tag{26}$$

The Eq.(25) together with Eq.(26) compose an equation system that can be represented in the following way

$$\begin{bmatrix} \mathbf{B} & -\Delta t\mathbf{G} \\ \mathbf{D} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{n} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc_1} \\ \mathbf{bc_2} \end{bmatrix} \tag{27}$$

The matrix $\mathbf{B}$ is given by

$$\mathbf{B} = \mathbf{M}_\rho - \frac{\Delta t}{Re}\mathbf{K} \tag{28}$$

The right side of the equations system (27) represents the variables known in time n, added the boundary conditions, that are the contributions of the known values of velocity and pressure.

$$\mathbf{r}^n = -\Delta t\left(-\frac{1}{Fr^2}\mathbf{M}_\rho\mathbf{g}\right) + \mathbf{M}_\rho\mathbf{u}_*^n \tag{29}$$

The method consists on decomposing the equations system (27) though a block fatoration. The work of (Lee et al., 2001) presented several ways of factoring such type of matrix, where each different factoring results on a new family of methods. By using a LU canonical block factoring, we obtain the following system

$$\begin{bmatrix} \mathbf{B} & \mathbf{0} \\ \mathbf{D} & \Delta t\mathbf{D}\mathbf{B^{-1}}\mathbf{G} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\Delta t\mathbf{B^{-1}}\mathbf{G} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{p}^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{r}^{n} \\ \mathbf{0} \end{bmatrix} + \begin{bmatrix} \mathbf{bc_1} \\ \mathbf{bc_2} \end{bmatrix} \tag{30}$$

The system as given in (30), if solved, results on the method known as Uzawa method (Chang et all., 2002). However, this method have an high computational cost, because of the need of inversion of the matrix $\mathbf{B}$ at each iteration. In this case, we used a process of approximations called Lumping for the inverse of the matrix $\mathbf{B}$. The new matrix is a diagonal matrix defined as the sum of the values of each line from the original matrix, storing the sum in the position of the element on the diagonal. Therefore, we have

$$\mathbf{B}\widetilde{\mathbf{u}} = \mathbf{r}^n + \mathbf{bc}_1 \tag{31}$$

$$\Delta t\mathbf{D}\mathbf{M}_\rho^{-1}\mathbf{G}\mathbf{p}^{n+1} = -\mathbf{D}\widetilde{\mathbf{u}} + \mathbf{bc}_2 \tag{32}$$

$$\mathbf{u}^{n+1} = \widetilde{\mathbf{u}} + \Delta t\mathbf{M}_\rho^{-1}\mathbf{G}\mathbf{p}^{n+1} \tag{33}$$

A procedure for the solution of the equations is given in the following order:

- Evaluate $\widetilde{\mathbf{u}}$ from Eq. (31);
- Evaluate $\mathbf{p}^{n+1}$ from Eq. (32);
- Evaluate the final velocity $\mathbf{u}^{n+1}$ using Eq. (33);
- Update the time step and continue until the final time or convergence are reached.

After update the components of the final velocity u^{n+1} and v^{n+1}, is necessary to update the component of the velocity w by using the equation of continuity, guaranteeing the condition of incompressibility.

4. NUMERICAL RESULTS

The method was validated for the case of stationary incompressible flow, producing nodal values close to the exact values. A solution that can be compared with the resulting solution of the simulations is the exact solution of the Poisson equation. The solution matches the corresponding velocity component in direction x of a stationary and developed flow over a square open duct with height $H = \frac{L}{2}$, i.e., the half of width $a = L$, considering that the surface is a symmetry line. The exact solution is given by

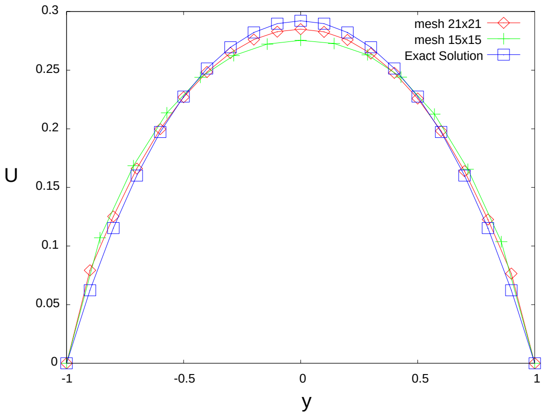
$$u(y,z) = \frac{\left(1-y^2\right)}{2} - \frac{16}{\pi^3}\sum_{k=1,k\ odd}^{\infty}\left\{\frac{sin\left(k\pi\left(1+y\right)/2\right)}{k^3 sinh\left(k\pi\right)} \times \left(sinh\left(k\pi\left(1+z\right)/2\right) + sinh\left(k\pi\left(1-z\right)/2\right)\right)\right\} \tag{34}$$

A square domain LxL for the surface, where $L = 2\ m$ was defined. In this case, no-slip conditions were imposed on the wall of the domain for the velocity components u, v and w. The pressure has zero value on the outflow of the duct.

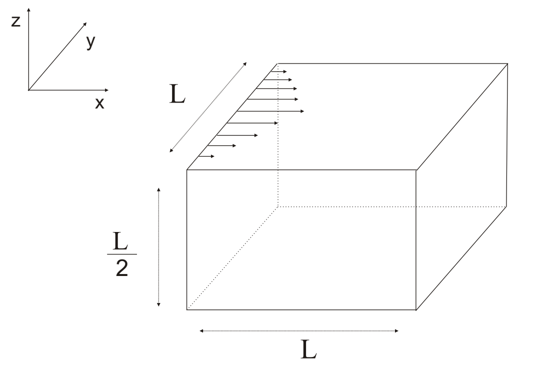
The velocity component in direction z has zero value in the upper and lower levels of the domain. The condition for fully developed flow defined at the inflow of the duct is given by the Eq. (34). The geometry of the domain is presented in the Figure 1(b), which shows the dimensions of the sides of the duct and the inflow of fluid.

When the stationary state is achieved, the values of the components of velocities along the duct are the same that those defined in the inflow of the duct. Then, one can compare the numerical results with the exact solution given by the Eq. (34). The model for this flow is detailed as follow. Dimension of domain: 2.0 m x 2.0 m x 1.0 m; Width of inflow: 2.0 m; Viscosity: 1.00 Ns/m^2; Density 1.0 kg/m^3; Mesh 1: 15x15x6 points, total of elements: 1960; Mesh 2: 21x21x6 points, total of elements: 4000, both equally distributed into five layers of elements, and Reynolds number: 10;

The Figure 1(a) shows the numerical result obtained by the simulation. The considered region when comparing the numerical result with the exact solution for this case is the outflow of the duct. From the figure, it can be seen that the more refined the mesh is, the more accurate the method is.



(a)



(b)

Figure 1. A) Comparison of obtained results with exact solution. B) Domain geometry.

In the simulations, the following boundary conditions were assigned. No-slip conditions were applied to the velocity components u, v e w on the walls of domain, including the steps regions. The value of pressure is zero at the outflow of the duct. In the inflow of the duct, we used $u = 1.0$ m/s, assigning zero value for v and w velocity components. The z velocity component has value of zero in the surface and the bottom of the domain. The boundary conditions used in the simulations can be better analized in the Figure 2.

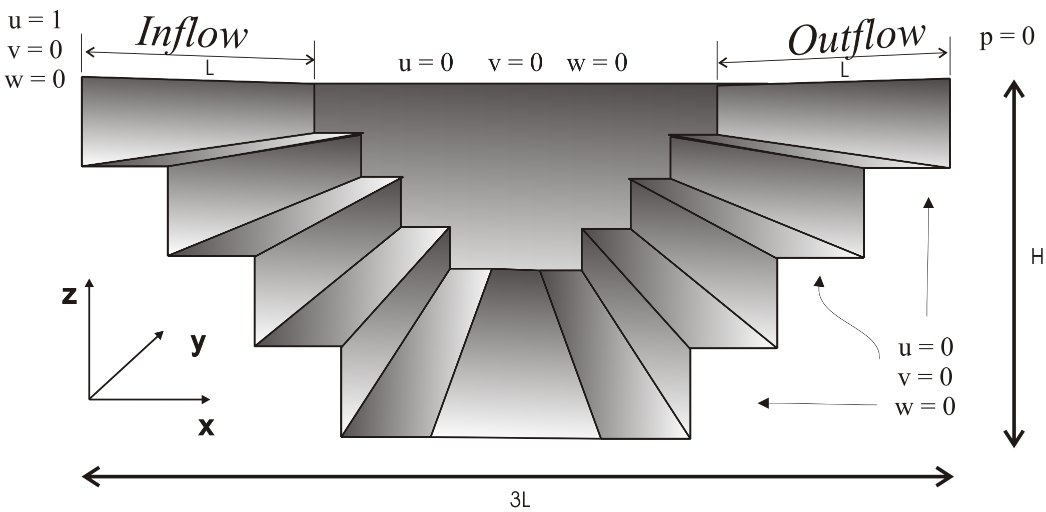


Figure 2. Perspective on y axis e boundary conditions for the simulation using stairs on reservoirs.

In this case was considered in simulations the height of the reservoir $H = 1.0$ m. Each step of the stair has height equal to the height of a prism element, length resulting of the combination of two prisms, as showed in Figure 3 and width equal to the combination of two prisms in the width of the duct.

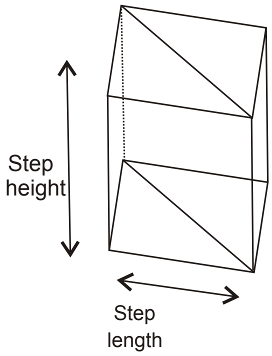


Figure 3. Detail of the dimension of a stair step.

The model for this fluid flow is: Domain dimension: 3.0 m x 1.0 m x 1.0 m; width of fluid inflow: 1.0 m; Viscosity: 1.00 Ns/m^2; Density: 1.0 kg/m^3; scale parameters: $L = 1.0$ m and $H = 1.0$ m; Reynolds number: 10 and 100.

The results presented in Figures 4 and 5 show the velocity component profiles in direction x obtained in the domain surface and at the outflow of the fluid flow for the Reynolds number 10 e 100. One can note that in the fluid flow, the velocity component in direction x has high values at the duct entrance, average values in the middle of duct and large values at the outflow of duct. This behavior is explained by the fact the domain present minor deepness at the beginning and at the ending of the duct, and major deepness at the middle of the duct, as can be observed in the Figure 2, a perspective of the domain.

The Figure 4 shows the velocity profiles obtained at the surface of the domain at the middle of the duct, using the Reynolds number 10 e 100, respectively.

The Figure 6 shows the number of iteration and the maximum residue obtained by the conjugate gradient method used at simulations for this case of stair steps on domain, by fixing five prisms layers.

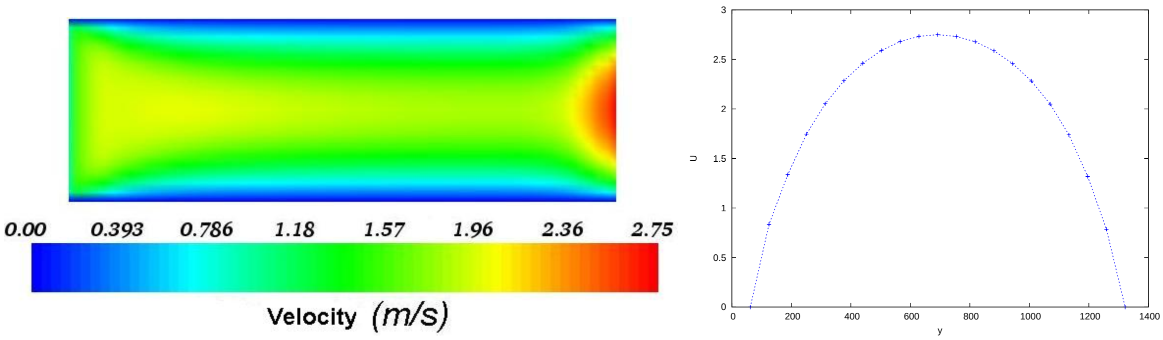


Figure 4. Simulation at a domain with stairs, field of velocity component in direction x with $Re = 10$, with five layer of prisms.

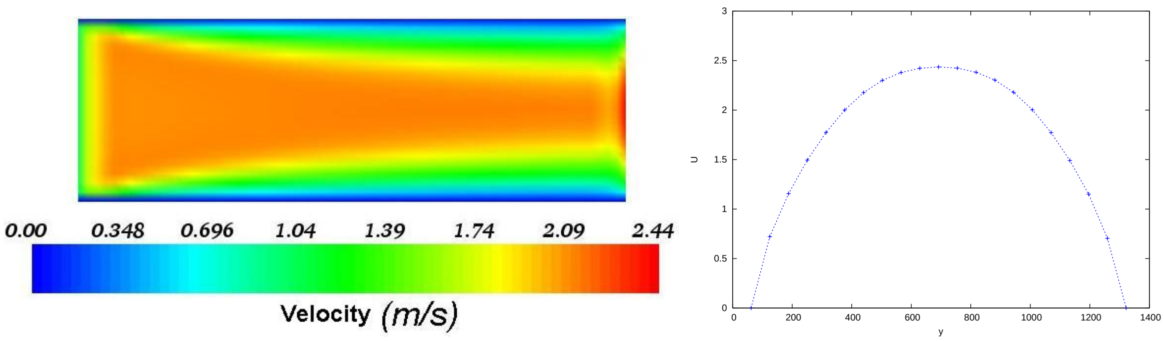


Figure 5. Simulation in a stair steps domain, field of velocity component in direction x with $Re = 100$, with five layer of prisms.

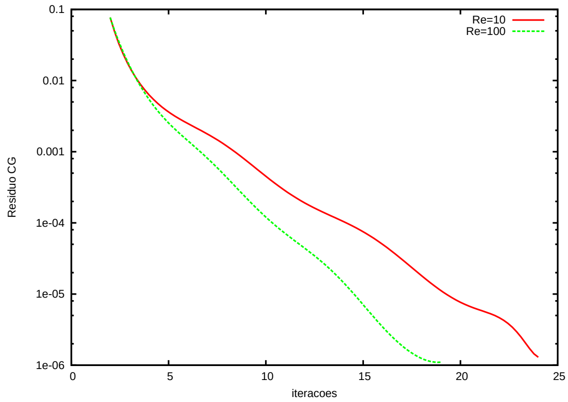


Figure 6. Residue of the conjugate gradient method over the iterations.

5. CONCLUSION

The goal of this work was the mathematical modeling and development of a method for the simulation of fluid flows in three-dimensional domains. The finite elements method was used to discretize the equations. The Projection method based on LU decomposition was used to extract the pressure component, and the using of Lumped matrices reduced the complexity of the algorithms, where the pressure gradient were calculated independently at each iteration. Then, the velocity value were corrected by the continuity equation, keeping the divergence field null. The solution of the linear systems was obtained by using the conjugate gradient method.

Future works about validation of the 3D simulation involve comparing the achieved results to real measurements obtained in controlled experiments and seeking similar works in this field that complement the analysis of the simulation itself.

6. ACKNOWLEDGEMENTS

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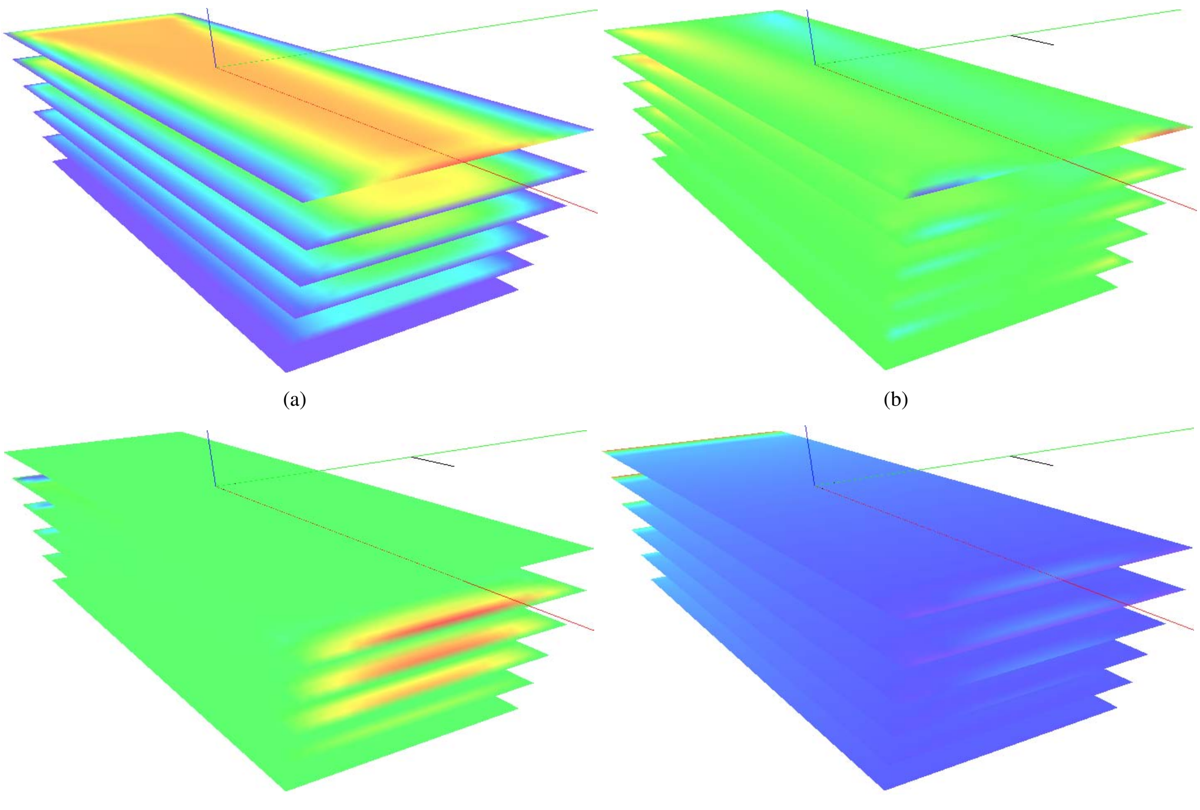
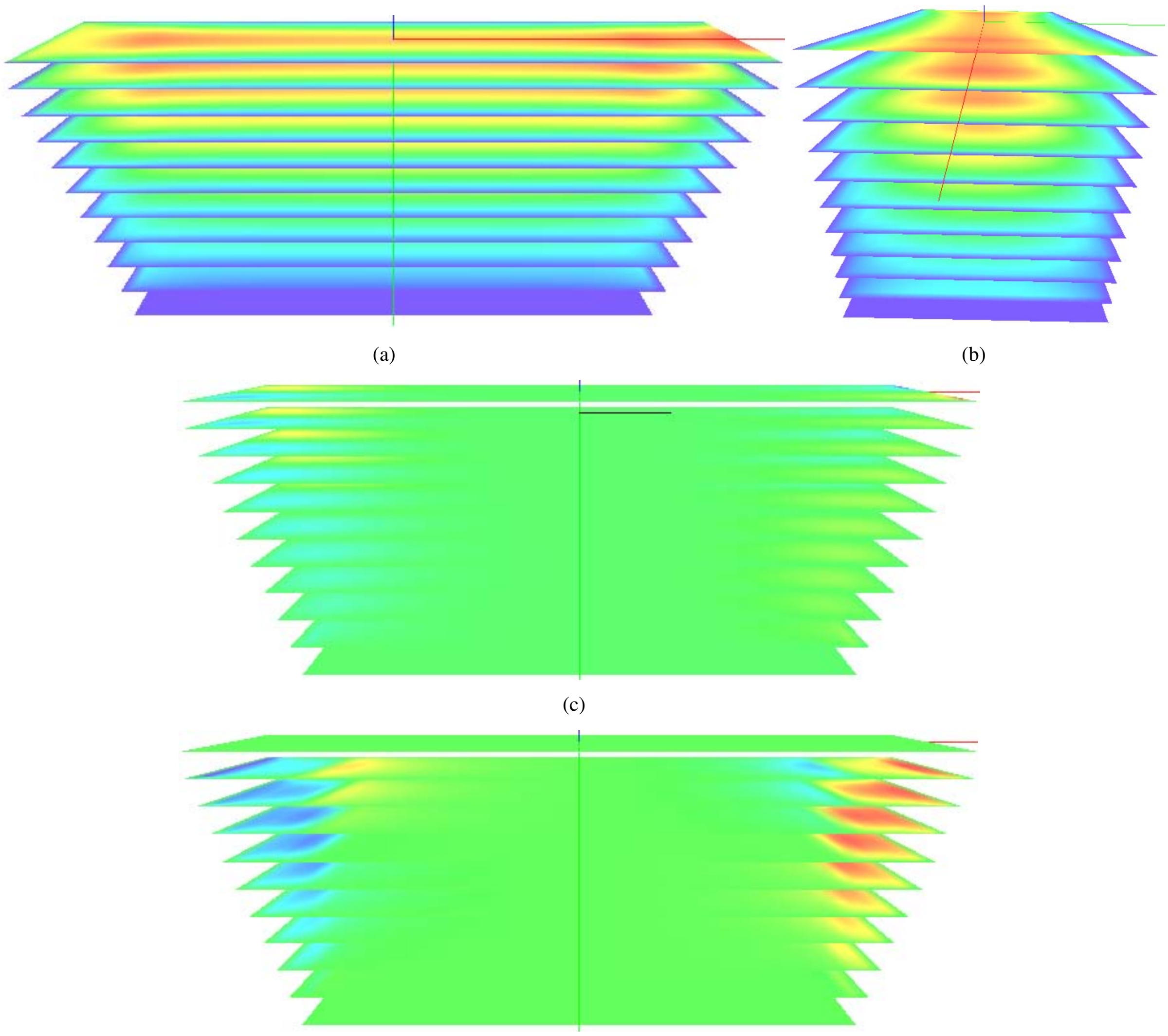


(a) (b) (c) (d)

Figure 7. Numerical simulation of the flow with stair on domain: a) Field of velocity component at direction x, b) Field of velocity component at direction y, c) Field of velocity component at direction z e d) Pressure profile using $Re = 100$.



(a) (b) (c) (d)

Figure 8. Numerical simulation of the flow with stair on domain: a) Field of velocity component at direction x, b) Field of velocity component at direction y, c) Field of velocity component at direction z e d) Pressure profile using $Re = 1$.