

# GAIN-SCHEDULED $\mathcal{L}_2$ CONTROL OF DISCRETE-TIME POLYTOPIC TIME-VARYING SYSTEMS

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**Abstract.** *This paper presents sufficient conditions that guarantee an upper bound  $\gamma$  on the  $\mathcal{L}_2$  gain of discrete-time Linear Parameter-Varying (LPV) systems. Moreover, synthesis conditions for gain-scheduled  $\mathcal{L}_2$  static output feedback controllers are also provided. The proposed stability and synthesis conditions are described in terms of linear matrix inequality (LMIs). LPV systems are systems whose dynamics changes according to a varying parameter, usually called the scheduling parameter. Practical examples of such systems are: aerospace structures that are constantly exposed to extreme variation of temperature and robotic systems commonly used in pick-and-place applications. In this work, it is assumed that the system matrices of the LPV model belong to a convex polytope. The stability condition, derived using Lyapunov theory, is described by an LMI which depends on the time-varying parameter and have to be satisfied at each time instant. This is an infinite dimensional problem which can not be solved numerically. To overcome this difficulty, a finite set of sufficient LMI conditions is derived by imposing on the Lyapunov matrix a polytopic structure. The proposed set of LMIs can take into account bounds on the rate of variation of the scheduling parameter. Thus, providing less conservative results than those obtained using stability conditions that allow the scheduling parameter to vary infinitely fast, as quadratic stability. Numerical simulations are performed to show the benefits of the proposed technique.*

**Keywords:** *discrete-time, linear time-varying system, gain-scheduled control, polytopic systems,  $\mathcal{L}_2$  gain.*

## 1. INTRODUCTION

Control design and identification of linear parameter varying (LPV) systems have received a lot of attention from the control community (Shamma and Athans, 1991; Apkarian and Adams, 1998; Scherer, 2001; De Caigny et al., 2009b). This stems from the fact that LPV models are useful to describe the dynamics of linear systems with time-varying parameters as well as to represent nonlinear systems in terms of a family of linear models (Rugh and Shamma, 2000; De Caigny et al., 2009d,e). In the LPV control framework, the scheduling parameters that govern the variation of the dynamics of the system are usually unknown, but supposed to be measured or estimated in real-time (Shamma and Athans, 1991; De Caigny et al., 2009c). There is a continuing effort towards the design of LPV controllers, scheduled as a function of the varying parameters, to achieve higher performance while still guaranteeing stability for all possible parameter variations (Apkarian and Adams, 1998; Scherer, 2001; De Caigny et al., 2009a).

Several analysis and synthesis results for LPV systems have been proposed in the literature based on different types of Lyapunov functions that are able to guarantee stability and performance. The appeal of Lyapunov theory comes from the fact that it allows to recast many analysis and synthesis problems as linear matrix inequality (LMI) optimization problems (Boyd et al., 1994). For LPV systems, the resulting parameter dependent LMI conditions need to be satisfied for the entire parameter space and, consequently, these LMI problems are infinite-dimensional. To arrive at a finite-dimensional set of LMI conditions, the choice of the parameterization (or the structure) of the Lyapunov matrix is essential.

Many of the existing Lyapunov approaches (Kaminer et al., 1993; Bernussou et al., 1989; Montagner et al., 2005b) use the notion of quadratic stability where the Lyapunov matrix is constant. This yields a finite set of LMIs that are usually conservative for practical applications, since it allows arbitrarily fast variation of the scheduling parameters. To alleviate some of the conservatism associated with the quadratic stability-based approaches, many works have proposed the use of parameter-dependent Lyapunov functions: for time-varying systems, piecewise Lyapunov matrices are considered by, amongst others, Leite and Peres (2004) and Amato et al. (2005), affine and polytopic structures are used in, for instance, Daafouz and Bernussou (2001); Montagner et al. (2005a); Oliveira and Peres (2008); De Caigny et al. (2008b,a).

The aim of this paper is to provide linear matrix inequality (LMI) conditions that enforces an upper bound on the  $\mathcal{L}_2$  gain for discrete-time linear systems with time-varying parameters belonging to a polytope with a prescribed bound on the rate of parameter variation. This paper also provide synthesis condition for gain-scheduled  $\mathcal{L}_2$  static output feedback controllers. The proposed bound on the rate of parameter variation is more conservative than the bound used in De Caigny

et al. (2008b). However, this new bound has a more realistic interpretation from a physical point of view.

This paper is organized as follows: Section 2 presents general theoretical background regarding  $\mathcal{L}_2$  gain of discrete-time LTV systems. Section 3 introduces some preliminaries with respect to the modeling of the uncertain domain, and then applies the results of Section 2 to the specific case of discrete-time polytopic LPV systems with known bounds on the rate of the parameter variation. Section 4 extends the analysis results and presents synthesis procedures for both gain scheduled and robust static output feedback controllers. A numerical example is presented in Section 5 that shows the benefits of the proposed approach. The conclusions and final remarks are presented in Section 6 and the Appendix presents the proof of main theorems.

### 1.1 Notation

The  $\mathcal{L}_2^n$  space of square-summable sequences on the set of nonnegative integers  $\mathbb{Z}_+$  is given by  $\mathcal{L}_2^n := \{f : \mathbb{Z}_+ \rightarrow \mathbb{R}^n : \sum_{k=0}^{\infty} f[k]^T f[k] < \infty\}$ . The corresponding 2-norm is defined as  $\|x[k]\|_2^2 = \sum_{k=0}^{\infty} x[k]^T x[k]$ . The identity matrix of size  $r \times r$  is denoted as  $I_r$ . The notation  $0_{n,m}$  indicates an  $n \times m$  matrix of zeros. The convex hull of a set  $X$  is denoted by  $\text{co}\{X\}$ .

## 2. $\mathcal{L}_2$ GAIN OF DISCRETE-TIME LTV SYSTEMS

Consider the following discrete-time linear time-varying (LTV) system

$$\begin{aligned} x[k+1] &= A[k]x[k] + B_w[k]w[k], & x[0] &= 0 \\ z[k] &= C_z[k]x[k] + D_w[k]w[k] \end{aligned} \quad (1)$$

where the state vector  $x[k] \in \mathbb{R}^n$ , the exogenous input  $w[k] \in \mathbb{R}^r$  and the system output  $z[k] \in \mathbb{R}^p$ . The system matrices are denoted by  $A[k] \in \mathbb{R}^{n \times n}$ ,  $B_w[k] \in \mathbb{R}^{n \times r}$ ,  $C_z[k] \in \mathbb{R}^{p \times n}$  and  $D_w[k] \in \mathbb{R}^{p \times r}$ .

The  $\mathcal{L}_2$  gain  $\gamma^*$  of system (1) is defined by the quantity  $\gamma^* = \sup_{\|w[k]\|_2 \neq 0} \|z[k]\|_2 / \|w[k]\|_2$ , with  $w[k] \in \mathcal{L}_2^m$  and  $z[k] \in \mathcal{L}_2^p$ . The next theorem provide an upper bound  $\gamma$  on the  $\mathcal{L}_2$  gain.

**Theorem 1** *If there exist symmetric positive-definite matrix  $P[k]$ , such that  $V(x[k], k) = x[k]^T P[k] x[k] > 0$  for all  $k \geq 0$  and*

$$\Delta V(x[k], k) + z[k]^T z[k] - \gamma^2 w[k]^T w[k] \leq 0, \quad \text{with} \quad \Delta V(x[k], k) = V(x[k+1], k+1) - V(x[k], k) \quad (2)$$

for all  $x[k]$  and  $z[k]$  satisfying system (1), then the  $\mathcal{L}_2$  gain is less than  $\gamma$ .

The proof of the Theorem 1 is presented in the Appendix. An upper bound on the  $\mathcal{L}_2$  gain of system (1) can be characterized using a parameter-dependent LMI as described in the next theorem.

**Theorem 2** *If there exist symmetric positive-definite matrix  $P[k]$ , such that*

$$\begin{bmatrix} P[k+1] & * & * & * \\ A[k]^T P[k+1] & P[k] & * & * \\ B_w[k]^T P[k+1] & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_z[k] & D_w[k] & I_p \end{bmatrix} \geq 0, \quad (3)$$

then  $\gamma$  is an upper bound on the  $\mathcal{L}_2$  gain of system (1).

The LMI condition in Theorem 2 are easily derived from Theorem 1. The proof is given in the Appendix.

## 3. $\mathcal{L}_2$ GAIN OF DISCRETE-TIME POLYTOPIC LPV SYSTEMS

In this section, Theorem 2 is particularized for the specific case of polytopic LPV systems. For this class of systems, it is provided a finite set of LMIs, defined in the vertices of a polytope, that guarantees an upper bound on the  $\mathcal{L}_2$  gain of the polytopic LPV system. Bounds on the rate of variation of the scheduling parameter are also considered. The modeling of the polytopic domain is first presented, afterwards, the finite sets of LMIs that guarantee an upper bound on the  $\mathcal{L}_2$  gain of the system are introduced.

### 3.1 Modeling of the Uncertainty Domain

Consider the following discrete-time polytopic time-varying system

$$\begin{aligned} x[k+1] &= A(\alpha[k])x[k] + B_u(\alpha[k])u[k] + B_w(\alpha[k])w[k], & x[0] &= 0 \\ z[k] &= C_z(\alpha[k])x[k] + D_u(\alpha[k])u[k] + D_w(\alpha[k])w[k] \end{aligned} \quad (4)$$

where the state vector  $x[k] \in \mathbb{R}^n$ , the exogenous input  $w[k] \in \mathbb{R}^r$ , the control input  $u[k] \in \mathbb{R}^m$  and the system output  $z[k] \in \mathbb{R}^p$ . The system matrices  $A(\alpha[k]) \in \mathbb{R}^{n \times n}$ ,  $B_w(\alpha[k]) \in \mathbb{R}^{n \times r}$ ,  $B_u(\alpha[k]) \in \mathbb{R}^{n \times m}$ ,  $C_z(\alpha[k]) \in \mathbb{R}^{p \times n}$ ,  $D_w(\alpha[k]) \in \mathbb{R}^{p \times r}$  and  $D_u(\alpha[k]) \in \mathbb{R}^{p \times m}$  belong to the polytope

$$\mathcal{P} = \left\{ (A, B_u, B_w, C_z, D_u, D_w)(\alpha[k]) : (A, B_u, B_w, C_z, D_u, D_w)(\alpha[k]) = \sum_{i=1}^N \alpha_i[k] (A, B_u, B_w, C_z, D_u, D_w)_i \right\}.$$

This model depends on  $\alpha[k] \in \Lambda_N$ , a vector of time-varying parameters lying in the unit simplex  $\Lambda_N$  for all  $k \geq 0$ , where

$$\Lambda_N = \{ \alpha \in \mathbb{R}^N : \sum_{i=1}^N \alpha_i = 1, \alpha_i \geq 0, i = 1, \dots, N \}. \quad (5)$$

The rate of the parameter variation is given by

$$\Delta \alpha_i[k] = \alpha[k+1] - \alpha[k], \quad i = 1, \dots, N. \quad (6)$$

Observe that due to (5) and (6), we have

$$\sum_{i=1}^N \Delta \alpha_i[k] = 0, \quad \forall k \geq 0. \quad (7)$$

It is assumed the rate of parameter variation  $\Delta \alpha[k]$  is limited by an *a priori* known bound  $b$  such that

$$-b \leq \Delta \alpha_i[k] \leq b, \quad i = 1, \dots, N \quad (8)$$

with  $b \in [0, 1]$ . Using similar ideas as in Oliveira and Peres (2008), the geometric aspects of the domain of the time-varying parameters are exploited to derive a model for the space where the vector  $\Delta \alpha[k]$  can lie. We now briefly present this model. The vector  $\Delta \alpha[k]$  is assumed to belong, for all  $k \geq 0$ , to the compact set

$$\Gamma_b = \{ \delta \in \mathbb{R}^N : \delta \in \text{co}\{h^1, \dots, h^M\}, \sum_{i=1}^N h_i^j = 0, j = 1, \dots, M \}.$$

From (7) and (8) the columns  $h^j, j = 1, \dots, M$  of the set  $\Gamma_b$  can be constructed as

$$[ h^1 \quad h^2 \quad \dots \quad h^M ] = b \begin{bmatrix} 1 & 1 & 1 & \dots & 0 \\ -1 & 0 & 0 & \dots & 0 \\ 0 & -1 & 0 & \dots & 0 \\ 0 & 0 & -1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & -1 \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix}$$

by construction, the number of columns  $M$  is given by  $M = N(N - 1)$ . Now, taking a convex combination of these  $M$  columns, we obtain

$$\Delta \alpha[k] = b \Gamma_b \beta[k], \quad \text{where } \beta[k] \in \Lambda_M.$$

Let  $\Gamma_j^{(i)}$  be the  $ij$ -element ( $i$ -th row and  $j$ -th column) of the  $N \times M$  matrix  $\Gamma_b$ , then  $\Delta \alpha_i[k]$  is given by

$$\Delta \alpha_i[k] = \sum_{j=1}^M b \Gamma_j^{(i)} \beta_j[k].$$

### 3.2 $\mathcal{L}_2$ Gain of Discrete-Time Polytopic LPV Systems

The LMI condition in Theorem 2 is now particularized for the polytopic systems (4) with  $u[k] = 0$ . This LMI condition follows directly from (3) in Theorem 2 by considering the specific time dependency of system (1) on the time-varying parameter  $\alpha[k]$ .

**Theorem 3** *If there exist parameter-dependent symmetric positive-definite matrix  $P(\alpha[k])$ , for all  $\alpha[k] \in \Lambda_N$ , such that*

$$\Phi(\alpha[k]) = \begin{bmatrix} P(\alpha[k+1]) & * & * & * \\ A(\alpha[k])^T P(\alpha[k+1]) & P(\alpha[k]) & * & * \\ B_w(\alpha[k])^T P(\alpha[k+1]) & 0_{m,n} & \gamma I_m & * \\ 0_{p,n} & C_z(\alpha[k]) & D_w(\alpha[k]) & I_p \end{bmatrix} \geq 0, \quad (9)$$

then  $\gamma$  is an upper bound on the  $\mathcal{L}_2$  gain of system (4).

The conditions of Theorem 3, which consist of evaluating the parameter-dependent LMI for all  $\alpha[k]$  in the unit simplex  $\Lambda_N$ , lead to an infinite dimensional problem. However, by imposing on the Lyapunov matrix  $P(\alpha[k])$  the affine parameter-dependent polytopic structure  $P(\alpha[k]) = \sum_{i=1}^N \alpha_i[k] P_i$ , a finite-dimensional set of LMIs in terms of the vertices of polytope  $\mathcal{P}$  can be obtained, as shown in the next theorem.

**Theorem 4** *If there exist symmetric positive-definite matrices  $P_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, N$  such that the following LMIs hold:*

$$\Phi_{il} = \begin{bmatrix} P_i + b\bar{P}_l & * & * & * \\ A_i^T(P_i + b\bar{P}_l) & P_i & * & * \\ B_{wi}^T(P_i + b\bar{P}_l) & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{zi} & D_{wi} & I_p \end{bmatrix} \geq 0 \quad (10)$$

with  $\bar{P}_l = \sum_{m=1}^N \Gamma_l^{(m)} P_m$  for  $l = 1, \dots, M$ ,  $i = 1, \dots, N$ , and

$$\Phi_{ijl} = \begin{bmatrix} P_i + P_j + 2b\bar{P}_l & * & * & * \\ A_i^T(P_j + b\bar{P}_l) + A_j^T(P_i + b\bar{P}_l) & P_i + P_j & * & * \\ B_{wi}^T(P_j + b\bar{P}_l) + B_{wj}^T(P_i + b\bar{P}_l) & 0_{m,n} & 2\gamma^2 I_m & * \\ 0_{p,n} & C_{zi} + C_{zj} & D_{wi} + D_{wj} & 2I_p \end{bmatrix} \geq 0 \quad (11)$$

with  $\bar{P}_l = \sum_{m=1}^N \Gamma_l^{(m)} P_m$  for  $l = 1, \dots, M$ ,  $i = 1, \dots, N-1$ ,  $j = i+1, \dots, N$ , then  $\gamma$  is an upper bound on the  $\mathcal{L}_2$  gain of system (4).

*Proof:* Multiply (10) by  $\alpha_i^2[k]\beta_l[k]$  and sum for  $l = 1, \dots, M$ ,  $i = 1, \dots, N$ . Multiply (11) by  $\alpha_i[k]\alpha_j[k]\beta_l[k]$  and sum for  $l = 1, \dots, M$ ,  $i = 1, \dots, N-1$ ,  $j = i+1, \dots, N$ . Summing the results yields

$$\Phi(\alpha[k]) = \sum_{l=1}^M \sum_{i=1}^N \alpha_i^2[k]\beta_l[k]\Phi_{il} + \sum_{l=1}^M \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i[k]\alpha_j[k]\beta_l[k]\Phi_{ijl}$$

The set of LMIs (10)-(11) guarantees that  $\Phi(\alpha[k])$  is positive semidefinite. ■

### 3.3 Extended $\mathcal{L}_2$ Gain of Discrete-Time Polytopic LPV Systems

It is possible to extend the previous  $\mathcal{L}_2$  gain conditions given by Theorem 3 using additional free variables. These extra variables will prove themselves valuable for control design. This is the context of Theorem 5.

**Theorem 5** *The  $\mathcal{L}_2$  gain of system (4) has as an upper bound  $\gamma$  if the following LMI is feasible*

$$\Psi = \begin{bmatrix} Q(\alpha[k+1]) & * & * & * \\ G(\alpha[k])^T A(\alpha[k])^T & G(\alpha[k]) + G(\alpha[k])^T - Q(\alpha[k]) & * & * \\ B_w(\alpha[k])^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_z(\alpha[k])G(\alpha[k]) & D_w(\alpha[k]) & I_p \end{bmatrix} \geq 0. \quad (12)$$

The proof can be found in De Caigny et al. (2008b). A computational finite-dimensional set of LMIs conditions are given in the next theorem.

**Theorem 6** *If there exist symmetric matrices  $Q_i \in \mathbb{R}^{n \times n}$  and matrices  $G_i \in \mathbb{R}^{n \times n}$ ,  $i = 1, \dots, N$  such that the following LMIs hold*

$$\Psi_{il} = \begin{bmatrix} Q_i + b\bar{Q}_l & * & * & * \\ G_i^T A_i^T & G_i + G_i^T - Q_i & * & * \\ B_{wi}^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{zi}G_j & D_{wi} & I_p \end{bmatrix} \geq 0$$

with  $\bar{Q}_l = \sum_{m=1}^N \Gamma_l^{(m)} Q_m$  for  $l = 1, \dots, M$ ,  $i = 1, \dots, N$ , and

$$\Psi_{ijl} = \begin{bmatrix} Q_i + Q_j + 2b\bar{Q}_l & * & * & * \\ G_i^T A_j^T + G_j^T A_i^T & G_i + G_j + G_i^T + G_j^T - Q_i - Q_j & * & * \\ B_{wi}^T + B_{wj}^T & 0_{m,n} & 2\gamma^2 I_m & * \\ 0_{p,n} & C_{zi} G_j + C_{zj} G_i & D_{wi} + D_{wj} & 2I_p \end{bmatrix} \geq 0$$

with  $\bar{Q}_l = \sum_{m=1}^N \Gamma_l^{(m)} Q_m$  for  $l = 1, \dots, M$ ,  $i = 1, \dots, N-1$ ,  $j = i+1, \dots, N$ , then  $\gamma$  is an upper bound on the  $\mathcal{L}_2$  gain of system (4).

The proof of Theorem 6 is similar to the proof of Theorem 4.

#### 4. $\mathcal{L}_2$ GAIN STATIC OUTPUT FEEDBACK

In this section, the analysis result presented in Theorem 6 is extended to provide a finite set of LMI conditions for the synthesis of robust and gain scheduled static output feedback controller that guarantee an upper bound on the closed-loop system  $\mathcal{L}_2$  gain for the discrete-time polytopic linear time-varying system (4).

##### 4.1 Gain Scheduled Case

We assume the first  $q$  states of the system can be measured in real-time for feedback without corruption by the exogenous input  $w[k]$  or the control input  $u[k]$ , that is,  $y[k] = C_y x[k]$  where  $y[k] \in \mathbb{R}^q$  is the measured output. The matrix  $C_y$  is assumed to have the structure

$$C_y = \begin{bmatrix} I_q & O_{q,n-q} \end{bmatrix}. \quad (13)$$

If this is not the case, one can use a similarity transformation as proposed in Geromel et al. (1996), whenever the output matrix is not affected by the time-varying parameter.

The aim is to provide a parameter-dependent control law  $u[k] = K(\alpha[k])y[k]$  with  $K(\alpha[k]) \in \mathbb{R}^{m \times q}$  such that the closed-loop system

$$\begin{aligned} x[k+1] &= A_{cl}(\alpha[k])x[k] + B_w(\alpha[k])w[k] \\ z[k] &= C_{cl}(\alpha[k])x[k] + D_w(\alpha[k])w[k] \end{aligned} \quad (14)$$

with

$$\begin{aligned} A_{cl}(\alpha[k]) &= A(\alpha[k]) + B_u(\alpha[k])K(\alpha[k])C_y \\ C_{cl}(\alpha[k]) &= C_z(\alpha[k]) + D_u(\alpha[k])K(\alpha[k])C_y \end{aligned}$$

is asymptotically stable with a guaranteed bound  $\gamma$  on the  $\mathcal{L}_2$  gain of the closed-loop system, from  $w[k]$  to  $z[k]$ , for all possible variation of the parameter  $\alpha[k] \in \Lambda_N$ .

The LMI from Theorem 5 is now applied to the closed-loop system matrices. By replacing the state-space matrices  $A(\alpha[k])$  and  $C_z(\alpha[k])$  from (12) with  $A_{cl}(\alpha[k])$  and  $C_{cl}(\alpha[k])$  from the closed-loop system (14), we obtain

$$\Theta(\alpha[k]) = \begin{bmatrix} Q(\alpha[k+1]) & * & * & * \\ \Theta_{21} & G(\alpha[k]) + G(\alpha[k])^T - Q(\alpha[k]) & * & * \\ B_w(\alpha[k])^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & (C_z(\alpha[k]) + D_u(\alpha[k])K(\alpha[k]))G(\alpha[k]) & D_w(\alpha[k]) & I_p \end{bmatrix} \geq 0$$

with  $\Theta_{21} = G(\alpha[k])^T (A(\alpha[k]) + B_u(\alpha[k])K(\alpha[k]))^T$ . Note that this matrix inequality is nonlinear due do the product of  $K(\alpha[k])$  and  $G(\alpha[k])$ . Now, applying the change of variable  $Z(\alpha[k]) = K(\alpha[k])G(\alpha[k])$ , we arrive at the following LMI

$$\Omega(\alpha[k]) = \begin{bmatrix} Q(\alpha[k+1]) & * & * & * \\ \Omega_{21} & G(\alpha[k]) + G(\alpha[k])^T - Q(\alpha[k]) & * & * \\ B_w(\alpha[k])^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_z(\alpha[k])G(\alpha[k]) + D_u(\alpha[k])Z(\alpha[k]) & D_w(\alpha[k]) & I_p \end{bmatrix} \geq 0$$

with  $\Omega_{21} = G(\alpha[k])^T A(\alpha[k])^T + Z(\alpha[k])^T B_u(\alpha[k])^T$ .

**Theorem 7** If there exist symmetric matrices  $Q_i \in \mathbb{R}^{n \times n}$ ,  $G_{i,1} \in \mathbb{R}^{q \times q}$ ,  $G_{i,2} \in \mathbb{R}^{n-q \times q}$ ,  $G_{i,3} \in \mathbb{R}^{n-q \times n-q}$  and  $Z_{i,1} \in \mathbb{R}^{m \times q}$ ,  $i = 1, \dots, N$  such that the following LMIs holds

$$\Omega_{il} = \begin{bmatrix} Q_i + b\bar{Q}_l & * & * & * \\ G_i^T A_i^T + Z_i^T B_{ui}^T & G_i + G_i^T - Q_i & * & * \\ B_{wi}^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{zi} G_i + D_{ui} Z_i & D_{wi} & I_p \end{bmatrix} \geq 0 \quad (15)$$

with  $\bar{Q}_l = \sum_{m=1}^N \Gamma_l^{(m)} Q_m$  for  $l = 1, \dots, M$ ,  $i = 1, \dots, N$ , and

$$\Omega_{ijl} = \begin{bmatrix} Q_i + Q_j + 2b\bar{Q}_l & * & * & * \\ G_i^T A_j^T + G_j^T A_i^T + Z_i^T B_{uj}^T + Z_j^T B_{ui}^T & G_i + G_j + G_i^T + G_j^T - Q_i - Q_j & * & * \\ B_{wi}^T + B_{wj}^T & 0_{m,n} & 2\gamma^2 I_m & * \\ 0_{p,n} & C_{zi} G_j + C_{zj} G_i + D_{ui} Z_j + D_{uj} Z_i & D_{wi} + D_{wj} & 2I_p \end{bmatrix} \geq 0 \quad (16)$$

with  $\bar{Q}_l = \sum_{m=1}^N \Gamma_l^{(m)} Q_m$  for  $l = 1, \dots, M$ ,  $i = 1, \dots, N-1$ ,  $j = i+1, \dots, N$ , with

$$G_i = \begin{bmatrix} G_{i,1} & 0_{q,n-q} \\ G_{i,2} & G_{i,3} \end{bmatrix} \quad \text{and} \quad Z_i = \begin{bmatrix} Z_{i,1} & 0_{m,n-q} \end{bmatrix} \quad (17)$$

are feasible, then the parameter-dependent controller

$$K(\alpha[k]) = Z(\alpha[k])G(\alpha[k])^{-1} \quad (18)$$

stabilizes the open loop system with  $\gamma$  is an upper bound on the  $\mathcal{L}_2$  gain of the closed-loop system (14).

*Proof:* Multiply (15) by  $\alpha_i^2[k]\beta_l[k]$  and sum for  $l = 1, \dots, M$ ,  $i = 1, \dots, N$ . Multiply (16) by  $\alpha_i[k]\alpha_j[k]\beta_l[k]$  and sum for  $l = 1, \dots, M$ ,  $i = 1, \dots, N-1$ ,  $j = i+1, \dots, N$ . Summing the results yields

$$\Omega(\alpha[k]) = \sum_{l=1}^M \sum_{i=1}^N \alpha_i^2[k]\beta_l[k]\Omega_{il} + \sum_{l=1}^M \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i[k]\alpha_j[k]\beta_l[k]\Omega_{ijl}$$

using (17) and (18) and considering the specific structure (13) for  $C_y$ , the LMI  $\Omega(\alpha[k])$  can be written as

$$\begin{bmatrix} Q(\alpha[k+1]) & * & * & * \\ G(\alpha[k])^T A_{cl}(\alpha[k])^T & G(\alpha[k]) + G(\alpha[k])^T - Q(\alpha[k]) & * & * \\ B_w(\alpha[k])^T & 0_{m,n} & \gamma^2 I_m & * \\ 0_{p,n} & C_{cl}(\alpha[k])G(\alpha[k]) & D_w(\alpha[k]) & I_p \end{bmatrix} \geq 0$$

Therefore, as a result of Theorem 5, feasibility of the LMIs (15) and (16) ensures that the closed-loop system (14) is asymptotically stable with an upper bound  $\gamma$  on its  $\mathcal{L}_2$  gain. ■

## 4.2 Robust Static Case

Robust  $\mathcal{L}_2$  gain static output feedback controller  $u[k] = KC_y x[k]$  is a particular case of gain-scheduled controller and can be calculated using theorem 7 by fixing  $K = ZG^{-1}$  with  $Z_i = Z$  and  $G_i = G$ .

## 5. NUMERICAL EXAMPLE

Consider the polytopic time-varying system (4) for  $n = 3$  and  $N = 2$  with the following system matrices:

$$[A_1 \ ; \ A_2] = \mu \begin{bmatrix} 1 & 0 & -2 & | & 0 & 0 & -1 \\ 2 & -1 & 1 & | & 1 & -1 & 0 \\ -1 & 1 & 0 & | & 0 & -2 & -1 \end{bmatrix}, [B_{w,1} \ ; \ B_{w,2}] = \begin{bmatrix} 0 & | & 0 \\ 1 & | & 0 \\ 0 & | & 1 \end{bmatrix}, B_{u,i} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, C_{z,i} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^T, D_{u,i} = D_{w,i} = 0,$$

with  $i = 1, 2$  and  $\mu > 0$ . These system matrices are borrowed from Oliveira and Peres (2008). The aim in this example is to determine the maximum bound  $b_{max}$  on the rate of parameter variation  $b$  as a function of the scalar  $\mu$  such that the system can be stabilized by an  $\mathcal{L}_2$  gain static output feedback controller. Both gain-scheduled and robust output feedback controllers are designed using the measurement equation  $y[k] = C_y x[k]$ , with all states available for feedback ( $C_y = I_n$ ). The proposed designs are compared with the results presented in De Caigny et al. (2008b).

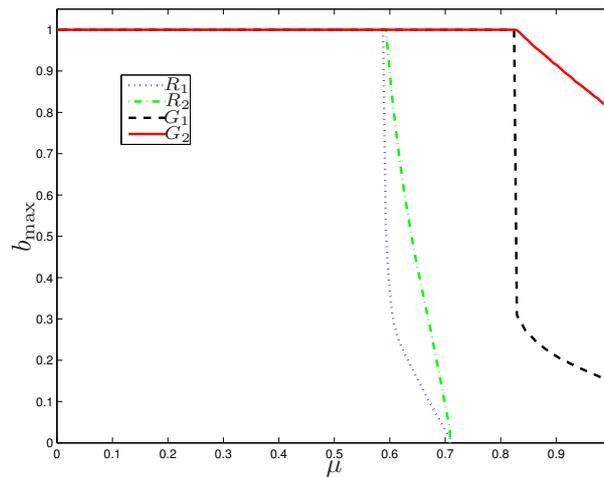


Figure 1. Maximal bound  $b_{max}$  on the rate of parameter variation as a function of the scalar  $\mu$ .

Figure 1 shows  $b_{max}$  as a function of  $\mu$ . The curves  $R_1$  and  $R_2$  are the robust controllers, and the curves  $G_1$  and  $G_2$  are the gain-scheduled controllers, with  $R_1$  and  $G_1$  denoting the controllers proposed in this paper and  $R_2$  and  $G_2$  denoting the controllers presented in De Caigny et al. (2008b).

For values of  $\mu \leq 0.5864$ , all control designs result in controllers that allow the parameters to vary arbitrarily fast in the unit simplex since  $b_{max} = 1$ . However, as  $\mu$  increases, the maximal allowed bound  $b_{max}$  becomes smaller. Obviously, this occurs first for the robust case  $R_1$ , since it is the most restrictive control design. Note also that since the gain-scheduled controllers are less restrictive than the robust controllers, the curves associated with the gain-scheduled controllers are always on the right of the curves associated with the corresponding (in terms of output measurements) robust controllers.

To check the achieved  $\mathcal{L}_2$  performance,  $\mu$  is now fixed to be  $\mu = 0.6185$ . Figure 2 shows the achieved upper bound  $\gamma$  on the closed-loop system  $\mathcal{L}_2$  gain as a function of the allowed bound  $0 \leq b \leq 1$  on the rate of parameter variation.

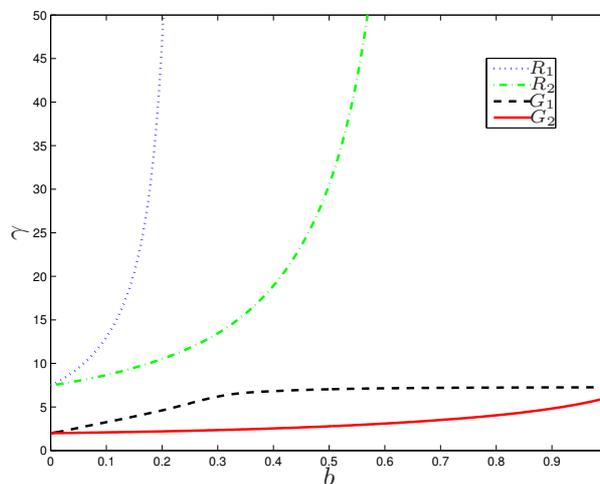


Figure 2. Guaranteed upper bound  $\gamma$  on the  $\mathcal{L}_2$  gain as a function of the scalar  $b$ .

For all control designs, it is clear from Figure 2, that as the bound  $b$  increases, the performance becomes worse since the upper bound  $\gamma$  increases. In the robust case  $R_1$  and  $R_2$ , the upper bound  $\gamma$  increases very fast as the value of the bound  $b$  increases. This can be expected since Figure 1 shows that for the robust case  $R_1$  with  $\mu = 0.6185$  the LMI conditions become infeasible for  $b > 0.2422$  and for the robust case  $R_2$  the LMIs become infeasible for  $b > 0.6836$ . In the gain scheduled case  $G_1$  and  $G_2$ , the conditions are feasible for all values of  $b$ .

For the specific case  $b = 1$ , where the parameters can vary arbitrarily fast in the unit simplex  $\Lambda_N$ , the gain-scheduled case  $G_1$  yields the gain  $\gamma = 7.2569$ , in case  $G_2$  the gain is  $\gamma = 6.0642$ . As seen in Figure 2, the LMI conditions in Theorem 7, by explicitly considering the bound  $b$  on the rate of variation, can provide a value very near  $\mathcal{L}_2$  gain bound  $\gamma$  for the gain-scheduled case  $G_1$  as compared to the results of the  $G_2$ , same more conservative. For the case  $b=0$ , the robust cases  $R_1$  and  $R_2$  presented same yields  $\gamma = 7.4563$  and gain-scheduled cases  $G_1$  and  $G_2$  presented yields  $\gamma = 1, 9935$ .

As shown in Figure 2, the results of the controllers  $G_1$  and  $G_2$  are near, but the results presented this paper with controller  $G_1$  are more reliable, by be more conservative.

## 6. CONCLUSION

In this work, new LMI conditions are presented for the synthesis of robust and gain-scheduled  $\mathcal{L}_2$  gain static output feedback controllers for discrete-time polytopic linear time-varying systems based on parameter dependent Lyapunov functions. The synthesis procedures explicitly take an a priori known bound on the rate of parameter variation into account, thus reducing the conservatism generally associated with methods that allow arbitrarily fast parameter variation.

Compared to the conditions in De Caigny et al. (2008b), the proposed approach yields similar results. They have different modeling for the rate of the parameter variation which has a more realistic physical interpretation.

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## APPENDIX

### Proof of Theorem 1

Summing Eq. (2) for  $k = 0, \dots, T$ , with  $x[0] = 0$ , gives

$$V(x[T+1], T+1) + \sum_{k=0}^T (z[k]^T z[k] - \gamma^2 w[k]^T w[k]) \leq 0.$$

Since  $V(x[T+1], T+1) > 0$ , this implies

$$\sum_{k=0}^T z[k]^T z[k] \leq \gamma^2 \sum_{k=0}^T w[k]^T w[k]$$

and consequently

$$\|z[k]\|_2 \leq \gamma \|w[k]\|_2.$$

Since Eq. (2) holds for all  $w[k] \in \mathcal{L}_2^m$  and  $z[k] \in \mathcal{L}_2^p$ , we conclude that

$$\sup_{\|w[k]\| \neq 0} \|z[k]\|_2 / \|w[k]\|_2 \leq \gamma. \quad \blacksquare$$

### Proof of Theorem 2

The LMI (3) using Schur complement is equivalent to

$$\begin{bmatrix} P[k] - A[k]^T P[k+1] A[k] - P[k] & -A[k]^T P[k+1] B_w[k] - C_z[k]^T D_w[k] & C_z[k]^T \\ -B_w[k]^T P[k+1] A[k] & -B_w[k]^T P[k+1] B_w[k] + \gamma^2 I & D_w[k]^T \\ C_z[k] & D_w[k] & I \end{bmatrix} \geq 0$$

using Schur complement again we have

$$\begin{bmatrix} P[k] - A[k]^T P[k+1] A[k] - C_z[k]^T C_z[k] & -A[k]^T P[k+1] B_w[k] - C_z[k]^T D_w[k] \\ -B_w[k]^T P[k+1] A[k] - D_w[k]^T C_z[k] & -B_w[k]^T P[k+1] B_w[k] - D_w[k]^T D_w[k] + \gamma^2 I \end{bmatrix} \geq 0$$

Now, this condition is satisfied if the following LMI holds

$$\begin{bmatrix} x \\ w \end{bmatrix}^T \begin{bmatrix} A[k]^T P[k+1] A[k] - P[k] + C_z[k]^T C_z[k] & A[k]^T P[k+1] B_w[k] + C_z[k]^T D_w[k] \\ B_w[k]^T P[k+1] A[k] + D_w[k]^T C_z[k] & B_w[k]^T P[k+1] B_w[k] + D_w[k]^T D_w[k] - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ w \end{bmatrix} \leq 0.$$

This last inequality is equivalent to

$$\Delta V(x[k], k) + z[k]^T z[k] - \gamma^2 w[k]^T w[k] \leq 0.$$

Thus, from Theorem 1, the  $\mathcal{L}_2$  gain is less than  $\gamma$ . \blacksquare

### Proof of Theorem 4

First note that if  $P(\alpha[k])$  is given by  $P(\alpha[k]) = \sum_{i=1}^N \alpha_i[k] P_i$ . Then  $P(\alpha[k+1])$  can be written as

$$\begin{aligned} P(\alpha[k+1]) &= \sum_{i=1}^N \alpha_i[k+1] P_i = \sum_{i=1}^N (\alpha_i[k] + \Delta \alpha_i[k]) P_i \\ &= \sum_{i=1}^N \alpha_i[k] P_i + \sum_{l=1}^N \Delta \alpha_l[k] P_l = \sum_{i=1}^N \alpha_i[k] P_i + \sum_{j=1}^M \sum_{l=1}^N b \Gamma_j^{(l)} \beta_j[k] P_l \end{aligned}$$

Define  $\bar{P}_j = \sum_{l=1}^N \Gamma_j^{(l)} P_l$ , we obtain

$$P(\alpha[k+1]) = \sum_{i=1}^N \alpha_i[k] P_i + \sum_{j=1}^M b \beta_j[k] \bar{P}_j.$$

We will only proof the (2,1)-element of  $\Phi(\alpha[k])$ . All the others follows similar steps. Note that

$$\Phi(\alpha[k])_{(2,1)} = A(\alpha[k])^T P(\alpha[k+1])$$

Now, using the above formula for  $P(\alpha[k+1])$ , we obtain

$$\Phi(\alpha[k])_{(2,1)} = \sum_{i=1}^N \alpha_i[k] A_i^T \left( \sum_{j=1}^N \alpha_j[k] P_j + \sum_{l=1}^M b\beta_l[k] \bar{P}_l \right)$$

that can be write (denoting  $\alpha[k]$  and  $\beta[k]$  by  $\alpha$  and  $\beta$ ) as

$$\Phi(\alpha)_{(2,1)} = \left( \sum_{i=1}^N \alpha_i A_i^T \right) \left( \sum_{j=1}^N \alpha_j P_j \right) + \left( \sum_{i=1}^N \alpha_i A_i^T \right) \left( \sum_{l=1}^M b\beta_l \bar{P}_l \right).$$

Multiplying the first term by  $\sum_{l=1}^M \beta_l = 1$  and the second term by  $\sum_{j=1}^N \alpha_j = 1$  we obtain

$$\Phi(\alpha)_{(2,1)} = \left( \sum_{i=1}^N \alpha_i A_i^T \right) \left( \sum_{j=1}^N \alpha_j P_j \right) \left( \sum_{l=1}^M \beta_l \right) + \left( \sum_{i=1}^N \alpha_i A_i^T \right) \left( \sum_{l=1}^M b\beta_l \bar{P}_l \right) \left( \sum_{j=1}^N \alpha_j \right).$$

The first term can be worked out as

$$\left( \sum_{i=1}^N \alpha_i^2 A_i^T P_i + \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j (A_i^T P_j + A_j^T P_i) \right) \left( \sum_{l=1}^M \beta_l \right)$$

and the second term can be worked out as

$$\left( \sum_{i=1}^N \alpha_i^2 A_i^T + \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j (A_i^T + A_j^T) \right) \left( \sum_{l=1}^M b\beta_l \bar{P}_l \right).$$

Summing these two up and rearranging the terms, we obtain

$$\Phi(\alpha)_{(2,1)} = \sum_{l=1}^M \sum_{i=1}^N \alpha_i^2 \beta_l A_i^T (P_i + b\bar{P}_l) + \sum_{l=1}^M \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j \beta_l (A_i^T (P_i + b\bar{P}_l) + A_j^T (P_j + 2b\bar{P}_l))$$

or equivalently as

$$\Phi(\alpha)_{(2,1)} = \sum_{l=1}^M \sum_{i=1}^N \alpha_i^2 \beta_l \Phi_{il(2,1)} + \sum_{l=1}^M \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j \beta_l \Phi_{ijl(2,1)}.$$

In this case  $\Phi(\alpha)$  can be written as

$$\Phi(\alpha) = \sum_{l=1}^M \sum_{i=1}^N \alpha_i^2 \beta_l \Phi_{il} + \sum_{l=1}^M \sum_{i=1}^{N-1} \sum_{j=i+1}^N \alpha_i \alpha_j \beta_l \Phi_{ijl}.$$