Analytical Solution to Stokes Problems with Wall Transpiration

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Abstract. *In this work, an analytical solution of the fluid behaviour over a flat plate with an impulsive and oscillating motion, starting from rest and with wall transpiration, is presented. The classical solution of this problem is given by Panton (1968) and is found to be an especial case of the solution found here. The analytical solution is obtained without the use of special transformations, like Laplace or Fourier transforms. Instead, an extension of the variable separation technique combined with similarity approach is used. The sine and cosine cases of Stokes' problems are treated by the same expression, by taking the real and imaginary part of the results. A non dimensional number, which can be stated as an Transpiration factor, is used to take into account the velocity of fluid injection by the wall. This parameter is of great influence in the velocity profiles.*

Keywords: *Stokes' problem, viscous fluid, oscillating wall, transient flow*

1. Introduction

The viscous flow of Newtonian fluids is described by the Navier-Stokes equations. These nonlinear partial differential equations form a very complex system, which exact solution can be found only for a small set of cases. Despite the fact that these exact solutions are limited to particular combinations of simple geometry and boundary conditions, they can provide a great knowledge or insight that can be helpful when dealing with more complex flow situations. On the other hand, exact solutions are very useful to check the accuracy of approximate methods such as numerical, asymptotic or even experimental ones.

One class of problem with known analytical solution are the Stokes' problems. These problems are related to the motion induced by an oscillating infinite plane wall in contact with a viscous fluid. The wall presents harmonic oscillations in direction parallel to itself. In the so called Stokes' first problem (Schlichting, 1979), the wall is initially at rest and a transient flow is induced to the fluid by the suddenly application of impulsive motion. In the Stokes' second problem the motion is generated by an oscillating plate. After some time, the transient motion vanishes and the fluid velocity at any point is just a harmonic oscillation with same frequency of the wall. This last problem was solved by Stokes (1851) and the solution of the starting part of the problem, in closed form and in terms of tabulated functions, was given by Panton (1968), which considered that the solution to this problem should be a summation of a transient and a steady state parts. Later, Erdogan (2000) has solved this problem using a Laplace transform technique. Using the same technique, Liu and Liu (2006) has calculated the solution of the extended Stokes problem, where the fluid has a finite depth. Finally, Erdogan and Imrak (2009) calculated the solution using the Fourier Transform technique.

A more general solution of the Stokes' problems can be derived when the fluid injection or suction at the plate surface is considered. The governing equation is augmented by a new term representing the momentum introduced into the flow by the transpiration of fluid. This solution has significant application in industry manufacturing, aeronautical systems, boundary layer control, chemical and mechanical engineering. This solution can also be applied to a more general problem, where the wall velocity is an arbitrary function, by using a Fourier series to represent this function and solving a sequence of Stokes' problems.

In this paper, the analytical solution of the starting Stokes' problems with wall fluid injection or suction is presented. To the best of authors knowledge, this the first closed form solution of this problem. The solution presented here, when no injection is applied, has the same form as the one devised by Panton (1968). The solution is calculated without the use of space transformation, instead, a modified version of the variable separation technique is evoked. The final form uses complementary complex error function. The solution when the plate has a impulsive start with wall transpiration is also presented and it results to be a specific case of Stokes' transient solution.

The outline of the rest of this paper is as follow: in the second section the basic equation for the Stokes' transient and steady-state problems are shown. In the third section, the analytical solutions are developed. Next, some results are presented and physical interpretations are given. Finally, some conclusions are drawn.

2. Basic Equations

The Stokes' problem considered here is stated as follow: consider a fluid with viscosity ν, initially at rest, occupying a half plane $y \geq 0$ and bounded on x-axis by an infinite plane wall. At time $t > 0$ the wall moves in x-direction with velocity given by $u_w(t)$. The fluid velocity $u \equiv u(y, t)$ is described by the Navier-Stokes equation which can be, considering the effects of fluid injection at the wall, reduced to

$$\frac{\partial u}{\partial t} + V_w \frac{\partial u}{\partial y} - \nu \frac{\partial^2 u}{\partial y^2} = 0 \qquad \text{for } y > 0 \text{ and } t > 0 \tag{1}$$

With boundary and initial conditions

$$u = u_w(t) = u_0 \exp(i\omega t) \qquad \text{at } y = 0, t > 0 \tag{2}$$
$$u = 0 \qquad \text{at } y \to \infty, t > 0 \tag{3}$$
$$u = 0 \qquad \text{at } y = 0, t = 0 \tag{4}$$

Where u_0 is maximum amplitude of wall velocity oscillation, ω is frequency of the wall velocity and $i = \sqrt{-1}$ is the imaginary constant. By using the wall velocity given in expression (2), the sine and cosine oscillation can be treated by taking the real and imaginary parts of velocity field. It should be noted that cosine oscillation presents a discontinuity at $t = 0$, when the wall velocity jumps from zero to u_0, differently from the sine function, which represents a more realistic situation. However, the mathematical treatment presented is the solution technique will treat these two cases effortlessly. We can use the set of non dimensional variables

$$U = \frac{u}{u_0} \qquad \tau = \omega t \qquad \eta = y \left(\frac{\omega}{\nu}\right)^{1/2} \qquad \xi = V_w/\sqrt{4\omega\nu} \tag{5}$$

to transform eq. (1) in

$$\frac{\partial U}{\partial \tau} + 2\xi \frac{\partial U}{\partial \eta} - \frac{\partial^2 U}{\partial \eta^2} = 0 \qquad \text{for } \eta > 0 \text{ and } \tau > 0 \tag{6}$$

and the boundary conditions (2)-(4) results

$$U = \exp(i\tau) \qquad \text{at } \eta = 0, \tau > 0 \tag{7}$$
$$U = 0 \qquad \text{at } \eta \to \infty, \tau > 0 \tag{8}$$
$$U = 0 \qquad \text{at } \eta = 0, \tau = 0 \tag{9}$$

Equation (1) is the classic Stokes' problem with fluid injection. The closed form solution to this set of equations, without transpiration, was given by Panton (1968).

3. Solution Technique

3.1 Non transient solution

The non transient solution is found after the transient effect of impulsive start has vanished and the fluid experiments a harmonic motion with the same wall oscillation frequency. Later, when we consider the transient motion, this solution will be useful. As long as the solution of this problem is not available when fluid injection is present, a solution technique for this problem will be developed in this section. The idea here is to obtain an ordinary differential equation from eq. (12), by taking a linear combination of the independent variables. Assume that the horizontal velocity as a function of $\varphi = A\tau + B\eta$ or $U = U(\varphi)$ so we have

$$\frac{\partial U}{\partial \tau} = A\frac{dU}{d\varphi} \qquad \frac{\partial U}{\partial \eta} = B\frac{dU}{d\varphi} \qquad \text{and} \qquad \frac{\partial^2 U}{\partial \eta^2} = B^2 \frac{d^2 U}{d\varphi^2} \tag{10}$$

Inserting the above hypothesis in eq. (12), the following ordinary differential equation arises

$$\frac{A + 2\xi B}{B^2} \frac{dU}{d\varphi} = B^2 \frac{d^2 U}{d\varphi^2} \tag{11}$$

The solution to this equation is

$$U(\varphi) = C_1 \frac{B^2}{A + 2\xi B} \exp\left(\frac{A + 2\xi B}{B^2}\varphi\right) + C_2 \tag{12}$$

The constants can be determined by using the boundary conditions

$$U(\tau, \eta = 0) = U(\varphi = A\tau) = e^{i\tau} \tag{13}$$
$$U(\tau, \eta \to \infty) = U(\varphi = B\eta) = 0 \tag{14}$$

Thus we have $C_1 = \frac{A+2\xi B}{B^2}$, $C_2 = 0$ and $A^2 + 2\xi BA - iB^2 = 0$. From this last equation, a relation between the constants A and B can be properly set as

$$B = i\left(-\xi \pm \sqrt{\xi^2 + i}\right) A \tag{15}$$

Then, the following simplification can be obtained:

$$U(\tau,\eta) = \exp\left[i\left(\tau + \frac{B}{A}\eta\right)\right] \tag{16}$$

It is necessary to point out that to satisfy the second boundary condition in eq. (14), we must have

$$Re\left[i\frac{B}{A}\right] \leq 0 \Rightarrow Re\left[-\xi \pm \sqrt{\xi^2+i}\right] \geq 0 \tag{17}$$

Where $Re(z)$ is the real part of complex argument z. The parameter ξ can be either positive (injection) or negative (suction), but the term $\sqrt{\xi^2+i}$ is strictly positive. The only way to maintain condition (17) is taking the positive sign at expression (15). Then, the final solution to the non transient Stokes' second problem with transpiration can be written as

$$U = \exp\left[i\,\tau + \eta\left(\xi - \sqrt{\xi^2+i}\right)\right] \tag{18}$$

When no injection is present $\xi = 0$ we have

$$U = \exp\left(-\frac{\eta}{\sqrt{2}}\right)\left[\cos\left(\tau - \frac{\eta}{\sqrt{2}}\right) + i\sin\left(\tau - \frac{\eta}{\sqrt{2}}\right)\right] \tag{19}$$

Note that eq. (19) is the exact solution to non transient problem found by Stokes (1851).

3.2 Transient solution

Assuming the horizontal velocity as a function $U = e^{\varphi}F(\tau,\eta)$ with $\varphi = A\tau + B\eta$, eq. (6) can be restated as

$$\frac{\partial F}{\partial \tau} + 2\left(\xi - B\right)\frac{\partial F}{\partial \eta} + F\left(A + 2\xi B - B^2\right) = \frac{\partial^2 F}{\partial \eta^2} \tag{20}$$

Choosing A and B so $A + 2\xi B - B^2 = 0$, we have

$$\frac{\partial F}{\partial \tau} + 2\left(\xi - B\right)\frac{\partial F}{\partial \eta} = \frac{\partial^2 F}{\partial \eta^2} \tag{21}$$

Taking $F \equiv F(\kappa)$ as a function of κ with the decomposition $\kappa = \left(\tilde{A}\tau + \tilde{B}\eta\right)g(\tau)$ and inserting into eq. (21), we arrive at the following ordinary differential equation

$$\frac{\left(\tilde{A}\tau + \tilde{B}\eta\right)g' + \left(\tilde{A} + 2\xi\tilde{B} - 2B\tilde{B}\right)g}{\tilde{B}^2 g^2}\frac{dF}{d\kappa} = \frac{d^2F}{d\kappa^2} \tag{22}$$

For expression (22) remains a ordinary differential equation, the term multiplying $\frac{dF}{d\kappa}$ must be a constant or a function of κ. If the later situation is considered, then the parameter g must also be a constant which reduces the solution of eq. (22) to the non transient case. Since the initial boundary condition imposes that the velocity profile must be at rest for $t = 0$, it means that the parameter g can not be a constant, otherwise, the velocity profile would not independent of y at $t = 0$, thus the following equation can be obtained:

$$\frac{1}{\tilde{B}^2 g^2}\left(\tilde{A}g'\tau + \left(\tilde{A} + 2\xi\tilde{B} - 2B\tilde{B}\right)g\right) + \frac{\tilde{B}g'}{\tilde{B}^2 g^2}\eta = C_0\kappa = C_0\left(\tilde{A}\tau + \tilde{B}\eta\right)g \tag{23}$$

This can be splited as

$$\frac{\tilde{B}g'}{\tilde{B}^2 g^2} = C_0\tilde{B}g \tag{24}$$

$$\frac{\tilde{A}g'\tau + \left(\tilde{A} + 2\xi\tilde{B} - 2B\tilde{B}\right)g}{\tilde{B}^2 g^2} = C_0\tilde{A}g\tau \tag{25}$$

From eq. (24) the first part we obtain a differential equation for $g(\tau)$ which has the solution $g = \pm i\,(\tilde{B}\sqrt{2C_0\tau})^{-1}$. Using this expression in second part we obtain $\tilde{A} = 2\tilde{B}\left(\xi - B\right)$. Then, the following equation can be written

$$\kappa = \frac{\pm i}{\sqrt{2C_0\tau}}\left[2\left(\xi - B\right)\tau + \eta\right] \tag{26}$$

The solution of eq. (22) is now given by (note that $0 < \kappa < \infty$)

$$F = C_3 \text{erfi}\left(\kappa\sqrt{C_0/2}\right) + C_2 \tag{27}$$

It is also useful apply the relation $\text{erfi}(z) = -i\ \text{erf}(i\,z) = -i\ (1 - \text{erfc}\, i\, z)$ to obtain

$$F = C_4\ \text{erfc}\left(i\,\kappa\sqrt{C_0/2}\right) + C_5 \tag{28}$$

And $U = e^{\varphi}F(\tau,\eta) = C_4\exp(\varphi)\,\text{erfc}\left(i\kappa\sqrt{C_0/2}\right) + C_5$. Using eq. (26)

$$U = C_4\exp(A\tau + B\eta)\left(\text{erfc}\left[\frac{\pm 1}{2\sqrt{\tau}}\left(2\left(\xi - B\right)\tau + \eta\right)\right]\right) + C_5 \tag{29}$$

But $U = 0$ at $\eta \to \infty$, then the positive option must must be considered in the above solution

$$U = C_4\exp(A\tau + B\eta)\,\text{erfc}\left[\frac{1}{2\sqrt{\tau}}\left(2\left(\xi - B\right)\tau + \eta\right)\right] + C_5 \tag{30}$$

To determine the remaining constants it is important to note that taking the limit $\tau \to \infty$, $\text{erfc}\left[\frac{1}{2\sqrt{\tau}}\left(2\left(\xi - B\right)\tau + \eta\right)\right] \to$ 2, then

$$U = 2C_4\exp(A\tau + B\eta) + C_5 \tag{31}$$

Equation (31) must have the same form of the steady state solution given by eq.(18), so the constants can be determined as $C_4 = 1/2;\ C_5 = 0;\ A = i$ and $B = \xi \pm \sqrt{\xi^2 + i}$. As the function erfc is bounded when $\eta \to \infty$ the two roots of expression (15) can be used as follow

$$U = \frac{1}{2}\exp\left[i\,\tau + \eta\left(\xi \pm \sqrt{\xi^2 + i}\right)\right]\text{erfc}\left[\sqrt{\tau}\left(\pm\sqrt{\xi^2 + i}\right) + \frac{\eta}{2\sqrt{\tau}}\right] \tag{32}$$

It can be written as

$$\begin{aligned} U = {} & \frac{1}{2}\exp\left[i\tau + \eta\left(\xi + \sqrt{\xi^2 + i}\right)\right]\text{erfc}\left[\sqrt{\tau}\sqrt{\xi^2 + i} + \frac{\eta}{2\sqrt{\tau}}\right] + \\ & \frac{1}{2}\exp\left[i\tau + \eta\left(\xi - \sqrt{\xi^2 + i}\right)\right]\text{erfc}\left[-\sqrt{\tau}\sqrt{\xi^2 + i} + \frac{\eta}{2\sqrt{\tau}}\right] \end{aligned} \tag{33}$$

The solution above is, as the best of authors' knowledge, the first known solution of the transient Stokes' problem when injection is present. When no injection is present $\xi = 0$ eq. (33) results

$$\begin{aligned} U = {} & \frac{1}{2}\exp\left[\frac{-\eta}{\sqrt{2}}\right]\left(\cos\left[\tau - \frac{\eta}{\sqrt{2}}\right] + i\ \sin\left[\tau - \frac{\eta}{\sqrt{2}}\right]\right) erfc\left[\frac{\eta}{2\sqrt{\tau}} - C\sqrt{\frac{\tau}{2}}\right] \\ & \frac{1}{2}\exp\left[\frac{\eta}{\sqrt{2}}\right]\left(\cos\left[\tau + \frac{\eta}{\sqrt{2}}\right] + i\ \sin\left[\tau + \frac{\eta}{\sqrt{2}}\right]\right) erfc\left[\frac{\eta}{2\sqrt{\tau}} + C\sqrt{\frac{\tau}{2}}\right] \end{aligned} \tag{34}$$

Where $C = 1 + i$. The expression above has the same form as the one devised by Erdogan (2000).

3.3 Impulsive start

The impulsive start (Stokes' first problem as shown in Schlichting (1979)) is the physical situation where the plate presents no oscillatory motion. In this case, the plate just moves with constant velocity in one direction parallel to itself.

This situation can also be seen as an especial case in eq. (33), when $\omega = 0$, we have $U = 1$. Restating this equation in dimensional form we have

$$U = \frac{1}{2}\exp\left(\frac{yV}{\nu}\right)\text{erfc}\left[\frac{V}{2}\sqrt{\frac{t}{\nu}} + \frac{y}{2\sqrt{t}}\right] + \frac{1}{2}\,\text{erfc}\left[\frac{V}{2}\sqrt{\frac{t}{\nu}} + \frac{y}{2\sqrt{t}}\right] \tag{35}$$

Using the nondimensional numbers $\zeta = \frac{y}{2\sqrt{t}}$ and $\lambda = \frac{V}{2}\sqrt{\frac{t}{\nu}}$, the previous equation can be written as

$$U = \frac{1}{2}\exp\left(4\lambda\,\zeta\right)\text{erfc}\left(\zeta + \lambda\right) + \frac{1}{2}\,\text{erfc}\left(\zeta - \lambda\right) \tag{36}$$

When no transpiration is present, $\lambda = 0$ and we obtain

$$U = \text{erfc}\left(\lambda\right) = 1 - \text{erf}\left(\frac{y}{2\sqrt{t}}\right) \tag{37}$$

Which is the same solution proposed by Stokes (1851).

4. Results and Discussions

Equation (33) is given in terms of function erfc. To evaluate this function using complex arguments, we used the definition given by Abramowitz and Stegun (1964). Some results from application of this equation are shown below. In figure (1) the transient profiles for the sine oscillation (in this case, we take the imaginary part of the expression), when no injection is present, are shown. These results are identical to the ones found by Panton (1968) and Erdogan (2000). The velocity profiles generated when the transpiration parameter is set as $\xi = 0.5$ are shown in fig. (2). It can be seen that the injection has a clearly effect in increasing the horizontal velocity magnitude. That should be expected as long as the fluid injected carries the particles near the wall deeper in to the fluid. When suction is present, as depicted in fig. (3), we should see the inverse effect: when the momentum transmitted to the fluid by the wall is taken away by the suction. As expected, the transient profile tends to the steady state when time parameter increases.

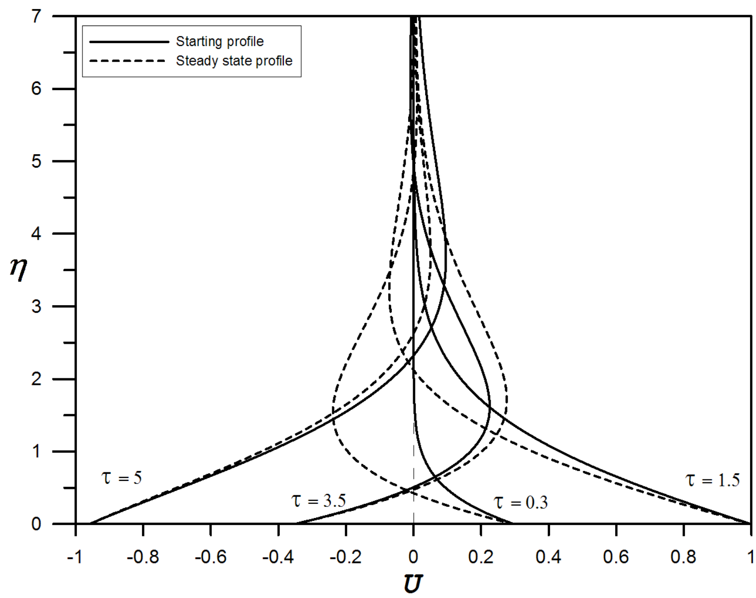

Figure 1. Horizontal velocity profile for various values of non dimensional time τ, using $\xi = 0$ (no injection) and a sine excitation of the wall.

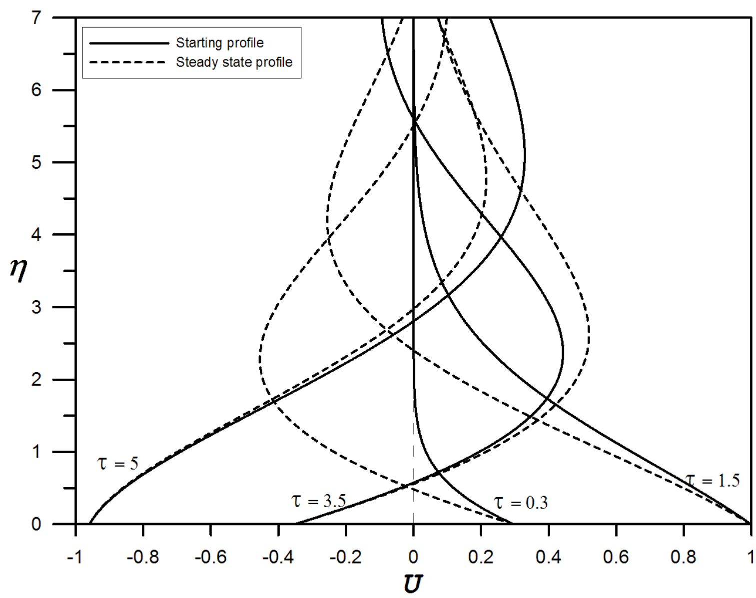

Figure 2. Horizontal velocity profile for various values of non dimensional time τ, using $\xi = 0.5$ (injection) and a sine excitation of the wall.

Figures (4) and (5) shows the transient and steady state velocity profiles for different values of transpiration parameter, for the cosine and sine excitation, respectively, at time $\tau = \pi/2$. It can be seen that the injection has a great impact in

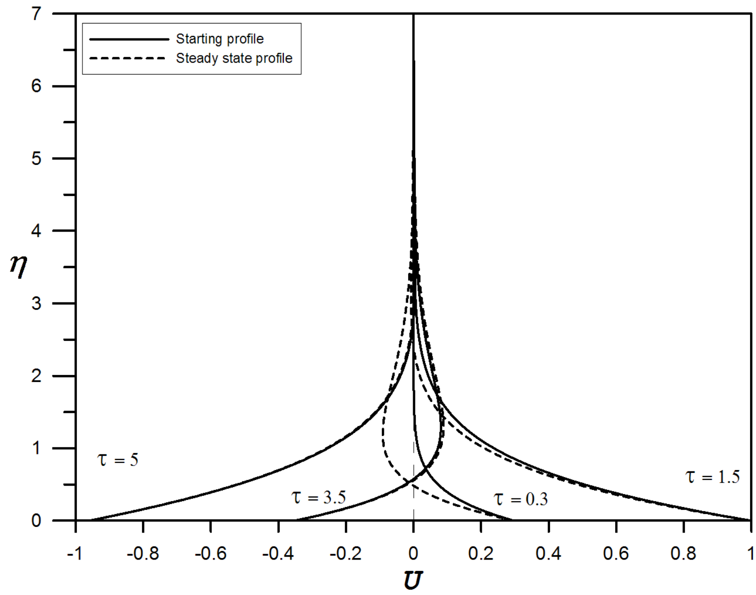

Figure 3. Horizontal velocity profile for various values of non dimensional time τ, using $\xi = -0.5$ (suction) and a sine excitation of the wall.

the transport of energy of the wall into the fluid. It can be also noticed that when then injection parameter increases, the transient effects takes a longer times to disappear.

Finally, fig. (6) shows the solution of eq. (36) for several values of injection parameter λ. As expected, this parameter has a great impact for energy propagation from the wall to fluid.

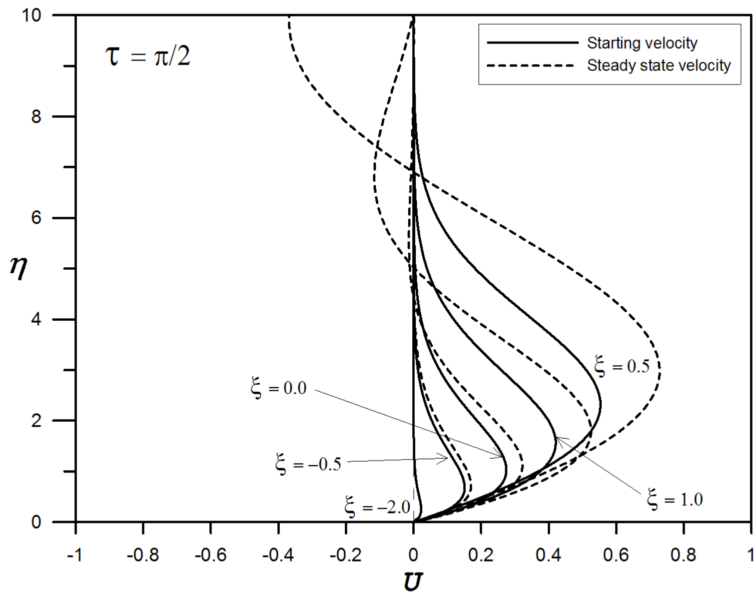

Figure 4. Horizontal velocity profile for various values of non dimensional time transpiration parameter ξ with a cosine excitation of the wall: transient velocity (full lines) and steady state velocity (dotted lines).

5. Final Remarks

In this work, the similarity method combined with an extended version of the variable separation technique was used to derive the solution of some moving boundary transient flow with wall transpiration. This approach seems to be very useful and can be used to obtain other analytical solutions for other flow situations, as was shown by Cruz et al. (2005). It was also shown that the effects of the fluid injection or suction, exerts a great influence on the flow pattern, by enhancing or suppressing the influence of the wall into the flow region. It is also important to mention that the above obtained solution can be used to derive some damping functions, that can be used to mimic the vanishing behavior of the turbulence stresses at the wall region.

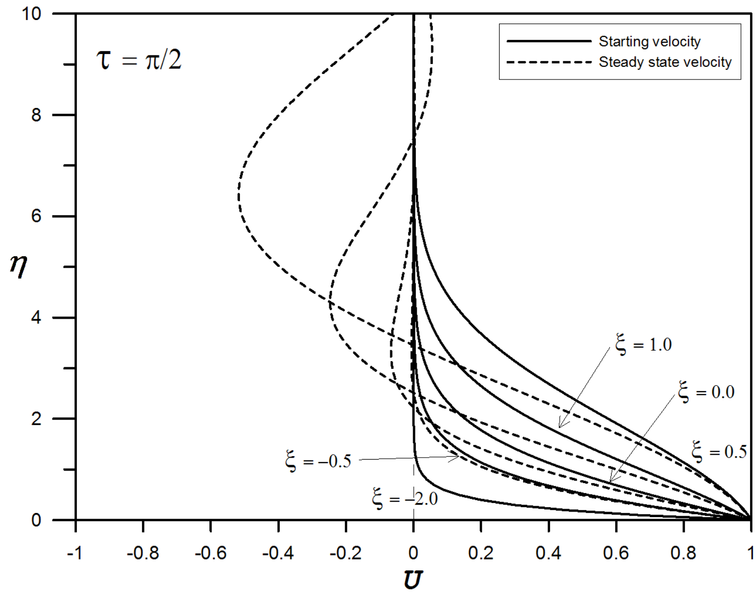

Figure 5. Horizontal velocity profile for various values of non dimensional time transpiration parameter ξ with a sine excitation of the wall: transient velocity (full lines) and steady state velocity (dotted lines).

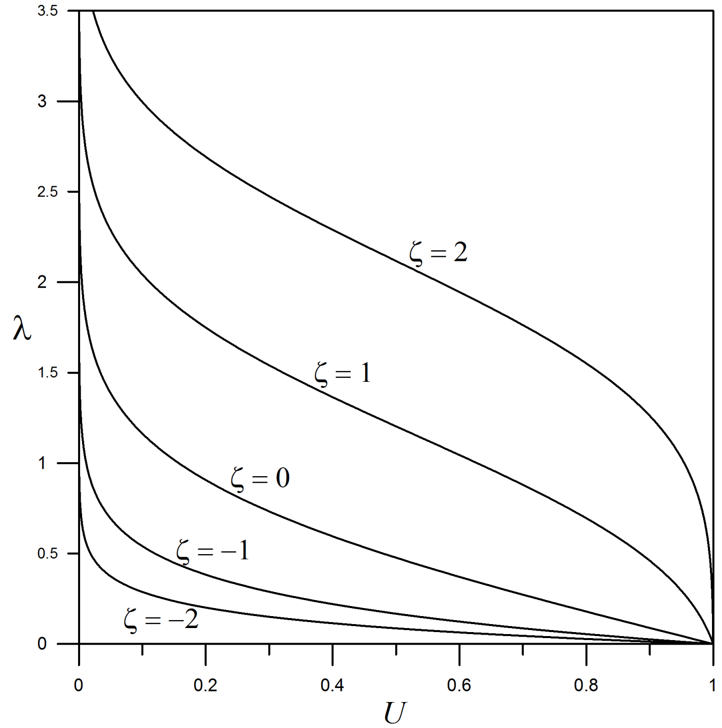

Figure 6. Velocity profile for various values of non dimensional parameter ζ with a impulsive start of the wall.

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