APPLICATION OF THE METHOD OF VORTEX DISTRIBUTION TO GENERATE AIRFOILS AND CASCADES

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Abstract. *This paper presents a method to calculate airfoils and cascades in potential flow based upon the distribution of vortices on the profile surface and develops a computational code and validates the predictions with available well accepted results. The technique is based upon Martensen (1959), Murugesan (1969) and Lewis (1991). Cascades are usually specified according to the application and they can be stationary or moving. The aerodynamic behavior and the aerodynamic characteristics of isolated foil are known and there are many techniques to calculate these foils such as the singularity method, conformal mapping and other technique where the singularity may be distributed over the camber line or on the surface of the foil. These analytic techniques can be extended to deal nicely with experimental foils such as the NACA´s. In the present paper, the method of distributing the vortices over the surface of the foil is in the reverse sense. Initially a distribution of velocity is specified at the leading edge stagnation point. The calculated velocity distribution over the surface of an initial arbitrary foil is compared with that required. The difference between them is used as an additional distribution of vortices imposed over the surface. A distribution of velocity component normal to the surface appears and this violates the concept of the stream line. The contour turns to be a streamline when its local inclination is adopted according to the ratio of the normal velocity and the required velocity. The resultant foil is reanalyzed in a direct form and the velocity distribution obtained is compared with the required distribution. The calculations are repeated until convergence is obtained. The resultant numerical code was applied to analytic symmetrical Joukowski foils and to experimental and asymmetrical foil NACA 65-1210. In all the cases the foils were generated using the vortex distribution technique and the resulting profiles were compared with the original profiles indicating a reasonably good agreement. Having showed that the proposed technique and the developed code are valid and produce good results, they are applied to analyse a cascade of NACA 65-1210 foils.*

***Keywords**: aerodynamics; cascades; vortex distribution; airfoils Joukowski; airfoils NACA*

1. INTRODUCTION

The use of cascades are very extensive in rotodynamic machines working with fluids and slurries, wind tunnels, blowers, ventilation units, air conditioning units and many other applications. A good and efficient pump, compressor or blower must have a good cascade design.

Some fifty years ago or more, cascade theories were not considered when dealing with small equipments because of the complexity of their calculations and hence the projects were realized using engineering experience and practical engineering data. These projects always suffered from low efficiency, bad functioning and maybe some additional serious problems such as flow induced vibrations, noise and cavitation when working with liquids. In recent years more attention is given to ambient comfort, high efficiency and low failure risk. This is aggravated when the equipment is crucial for safe operation as in gas turbines and compressors for aviation. These new tendencies, the need to have silent and efficient machines with low maintenance and high reliability urged the necessity for more research in the different areas to achieve a good quality project.

The essential part of rotodynamic machines is the cascade which converts energy from one form to another; it must be efficient, silent and as small as possible. These new requirements can only be achieved by a good design and hence cascade analysis.

The literature is rich with experimental and theoretical studies related to this area. The recent computational facilities led to a great deal of numerical studies on the different aspects of cascade design. Among these studies one mention Horlock's, Lewis' and others have very dense contributions to both the theoretical, numerical and experimental aspects of cascades.

This paper presents a method to calculate the geometry and aerodynamic characteristics of airfoils and cascades based upon surface distribution of singularities. The numerical code was applied to calculate Joukowski airfoils with and without angle of attack and to analyze a NACA cascade using the NACA 65-1210 airfoil.

2. FORMULATION

Martensen (1959) presented a solution of the two dimensional potential flow problem over a cascade. In his development he used the Green's function technique to obtain the stream line function ψ over a closed contour C . The differentiation of this function with respect to the normal to this contour combined with the Dirichlet boundary condition results

$$v(s_m) - \frac{1}{2\pi} \int_C K(s_m, s; t)\, v(s)\, ds = 2\, W_\infty \left(\cos\beta_\infty \left.\frac{\partial x}{\partial s}\right|_{s_m} + sen\beta_\infty \left.\frac{\partial y}{\partial s}\right|_{s_m} \right) \tag{1}$$

where s_m and s are distances measured around the profile. The coupling coefficient $K(s_m, s; t)$ in terms of the cascade pitch t is

$$K(s_m, s; t) = \frac{2\pi}{t} \frac{\frac{\partial y}{\partial s} senh\left[\frac{2\pi}{t}(x_m - x)\right] - \frac{\partial x}{\partial s} sen\left[\frac{2\pi}{t}(y_m - y)\right]}{\cosh\left[\frac{2\pi}{t}(x_m - x)\right] - \cos\left[\frac{2\pi}{t}(y_m - y)\right]}$$

and for an isolated profile it is

$$K(s_m, s; \infty) = 2 \frac{\frac{\partial y}{\partial s}(x_m - x) - \frac{\partial x}{\partial s}(y_m - y)}{(x_m - x)^2 + (y_m - y)^2}$$

where U_∞ and V_∞ are the components of the flow relative velocity W_∞ with inclination β_∞ relative the x-axis. For the isolated airfoil W_∞ is the non perturbed velocity far away from the airfoil W_1, while W_∞ for the cascade is the equivalent mean velocity defined as the mean vector average of the velocities far away from the cascade W_1 and W_2 .

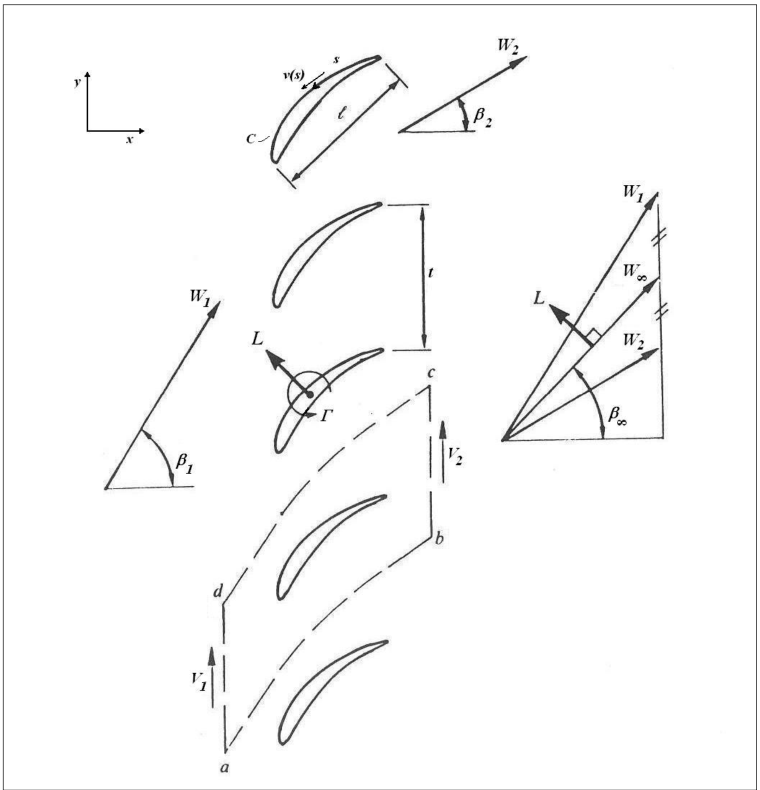

Figure 1. Geometry of cascade and triangles of velocity

Using the Dirichlet boundary condition and the properties of ψ one can obtain $\partial\psi/\partial n|_i = 0$ and

$$v = -\left.\frac{\partial \psi}{\partial n}\right|_e \tag{2}$$

In the solution of the potential flow around an isolated profile or cascade, the contour is substituted by a vortex sheet of strength $\gamma(s)$. The velocity of the fluid tangent to the surface of the body charges from zero inside the sheet to $v(s)$ outside the vortex sheet. From this one can verify that the coupling coefficient $k(s_m, s; t) = K(s_m, s; t)/(4\pi)$ has the same value as the velocity parallel to the surface of the body and induced by an element of vorticity $\gamma(s)ds$.

One parametric representation of the contour C can be $x = x(\phi)$ and $y = y(\phi)$, with $0 \le \phi \le 2\pi$ as the parameter. Using the transformation

$$\gamma(\phi)\, d\phi = \gamma(s)\, ds$$

one can obtain

$$\gamma(s) = \frac{\gamma(\phi)}{ds/d\phi} = \frac{\gamma(\phi)}{\sqrt{(x')^2 + (y')^2}} \tag{3}$$

where $x' = dx/d\phi$, $y' = dy/d\phi$. Hence Eq. (1) becomes

$$\frac{\gamma(\phi_m)}{W_\infty} - \frac{1}{2\pi}\int_0^{2\pi} K(\phi_m, \phi; t)\, \frac{\gamma(\phi)}{W_\infty}\, d\phi = 2\,(x'_m \cos\beta_\infty + y'_m sen\beta_\infty) \tag{4}$$

where $x'_m = x'(\phi = \phi_m)$ and $y'_m = y'(\phi = \phi_m)$.

The coupling coefficient for infinite cascade becomes

$$K(\phi_m, \phi; t) = \frac{2\pi}{t}\; \frac{y'_m senh\left[\frac{2\pi}{t}(x_m - x)\right] - x'_m sen\left[\frac{2\pi}{t}(y_m - y)\right]}{\cosh\left[\frac{2\pi}{t}(x_m - x)\right] - \cos\left[\frac{2\pi}{t}(y_m - y)\right]} \tag{5}$$

and for isolated profile becomes

$$K(\phi_m, \phi; \infty) = 2\; \frac{y'_m(x_m - x) - x'_m(y_m - y)}{(x_m - x)^2 + (y_m - y)^2} \tag{6}$$

One can observe that the coupling coefficient $K(\phi_m, \phi_m; t)$ associated with the self induced velocity by the element m is indeterminate, but can be treated by L'Hospital rule to be

$$K(\phi_m, \phi_m; t) = \frac{y''_m x'_m - x''_m y'_m}{{x'_m}^2 + {y'_m}^2} \tag{7}$$

The contour integral is evaluated by a trapezoidal rule with $2N$ elements of vortex γ_m where N elements lie on the upper surface and another N elements on the lower surface. The control points are located at the mid point of the elements $\phi_n = (n - 1/2)\pi/N$, $n = 1, 2, \ldots, 2N$. In this manner Eq. (4) becomes

$$\sum_{n=1}^{2N} K_{m,n}\, \gamma_n = -4N\,(x'_m \cos\beta_\infty + y'_m sen\beta_\infty) \tag{8}$$

where

$$\left.\begin{aligned} \gamma_n &= \frac{\gamma(\phi_n)}{W_\infty}, \\ K_{m,n} &= K(\phi_m,\phi_n;t), \qquad n \neq m \\ K_{m,m} &= K(\phi_m,\phi_n;t) - 2N, \quad n = m \\ b_m &= -4N\left(x'_m \cos\beta_\infty + y'_m sen\beta_\infty\right) \end{aligned}\right\} \tag{9}$$

Applying Eq. (8) in each control point one can obtain a system of $2N$ simultaneous linear equations in γ_m. After solving this system of equations the velocity $v(s_m)$ can be determined from

$$v(s_m) = \gamma(s_m) = \frac{\gamma_m W_\infty}{\sqrt{(x'_m)^2 + (y'_m)^2}}$$

But, since the matrix is singular, the solution of the system of equations is possible by applying the boundary condition due to Kutta at the trailing edge. The static pressure is the same on the both surfaces near the trailing edge, then the vortex of the two elements adjacent to this point must have the same absolute value, that is $\gamma(s_{2N}) = -\gamma(s_1)$.

The circulation generated by the airfoil is equal to the total strength of the contour vortex Γ obtained as

$$\Gamma = \sum_{m=1}^{2N} \gamma(\phi_m)\Delta\phi_m = W_\infty \sum_{m=1}^{2N} \gamma_m \Delta\phi_m \tag{10}$$

Based upon the circulation around the airfoil $\Gamma = t\,(V_2 - V_1)$, the continuity equation $U_1 = U_2 = U_\infty$ and the velocity triangles of Fig.1, one can write

$$\Gamma = t\, W_\infty \cos\beta_\infty \left(\tan\beta_2 - \tan\beta_1\right) \tag{11}$$

In the case of isolated profile, if $W_\infty = W_1$, $\beta_\infty = \beta_1$, given the geometry of the airfoil (x, y) and the conditions of the free stream at entry W_1 and β_1, and from Eq. (8), one can obtain the vortex distribution γ_m; from Eq. (10) the total circulation Γ and finally from Eq. (11) one can determine the exit angle of the flow β_2.

For a cascade, the surface distribution of vortex and the total circulation around each foil of the cascade are obtained in a similar manner if the cascade geometry is known and the average flow conditions equivalent to W_∞ and β_∞ are initially established. The entry angle β_1 and exit angle β_2 are calculated from Eq. (11) and the fact that $\tan\beta_\infty = (\tan\beta_1 + \tan\beta_2)/2$ are used to obtain

$$\beta_1 = \arctan\left[\tan\beta_\infty - \Gamma/(2\,t\,W_\infty \cos\beta_\infty)\right] \tag{12}$$

In practice, the design problem is different, the designer needs to know the exit conditions, that is, β_2 and W_2 for a range of entry conditions β_1 and W_1 for a given cascade geometry. This can be done following the general lines suggested by Lewis (1991).

Knowing the RHS of the system of equations, according to Eq. (8), one can divide the two independent parts in two systems and solve for unit flow velocities. The respective solutions γ_u and γ_v of the non dimensional strength of vortex gives the corresponding unit contour circulations Γ_u and Γ_v as below

$$\left.\begin{aligned} \Gamma_u &= \sum_{m=1}^{2N} \gamma_u(\phi_m)\Delta\phi_m = \sum_{m=1}^{2N} \gamma_{u_m} \Delta\phi_m \\ \Gamma_v &= \sum_{m=1}^{2N} \gamma_v(\phi_m)\Delta\phi_m = \sum_{m=1}^{2N} \gamma_{v_m} \Delta\phi_m \end{aligned}\right\}$$

The expression for the exit angle β_2 in terms of the unit circulation Γ_u and Γ_v is

$$\beta_2 = \arctan\left\{\left(\frac{1+\Gamma_v/2t}{1-\Gamma_v/2t}\right)\tan\beta_1 + \left(\frac{2}{1-\Gamma_v/2t}\right)\frac{\Gamma_u}{2t}\right\}$$

From the equation of momentum and the relations obtained from the velocity triangles one can conclude that the lift force is normal to the direction of the mean velocity W_∞. In this case the pressure coefficient is given by

$$C_{p_\infty} = \frac{p-p_1}{\frac{1}{2}\rho W_\infty^2} = 1-\left(\frac{\gamma(s)}{W_\infty}\right)^2 = 1-\left(\frac{\gamma(\phi)}{\frac{ds}{d\phi}W_\infty}\right)^2$$

The pressure coefficient can be given alternatively by

$$C_{p_1} = \frac{p-p_1}{\frac{1}{2}\rho W_1^2} = C_{p_\infty}\left(\frac{\cos\beta_1}{\cos\beta_\infty}\right)^2$$

3. CALCULATION PROCEDURE

The method used in this paper to design an isolated foil or a cascade of airfoils uses iteratively the method of surface vortex following the general lines of Murugesan and Railly (1969) and Lewis (1991). The general procedure steps adopted here are as follows.

(i) Specify initially a velocity distribution or vortex on the surface $v_R(s)$.
(ii) Choose as a first approximation a profile which can be ellipse or a standard airfoil.
(iii) Discretize this profile in $2N$ elements.
(iv) The velocity distribution on the contour is calculated.
(v) The frontal stagnation point is localized; calculate the contour length between the frontal stagnation point and the control point of each element and the velocity of this point is interpolated from the specified velocity distribution.
(vi) The calculated and the specified velocities are compared and the criterion of convergence is verified.
(vii) Assume an additional velocity distribution equal to the difference between the calculated and specified velocities.
(viii) Calculate the normal velocity distribution over the contour induced by the additional vortex sheet.
(ix) Determine the local tangent to the contour. The integration of the new inclination from the frontal stagnation point results in a new contour for each of the suction and pressure surfaces.
(x) If the trailing edge points calculated at the suction and pressure surfaces are not coincident, adopt a procedure to close the profile.

For this last profile repeat the calculations (iv) to (x) until the convergence limit (vi) is achieved.

4. APPLICATIONS

As an application of the proposed method of calculation four study cases will be presented. Initially two cases of the symmetric Joukowski airfoils will be used one without and other with angle of attack. In the third study case a NACA foil will be designed. The last application is for the analysis of a NACA cascade.

4.1. Design of the symmetric Joukowski airfoil

The Kutta-Joukowski transformation $\zeta = z + b^2/z$ is used. This transformation makes a circle of radius r_0 and center at the origin of the plane z to a symmetric airfoil in the transformed plane ζ. In this case the ratio b/r_0 is taken as $0{,}9333$ and the angle of attack is 0°. The foil velocity distribution determined analytically is used to compare with the numerical predictions.

Figure 2 shows the evolution of the calculated foil shapes during the calculation of a Joukowski symmetric airfoil with $2N = 160$ control points. As was commented before the initial adopted geometry is an ellipse. Figure 2 shows the numerical development of the profile during the calculations and the final profile obtained after 1977 iterations is the inner geometry in the figure. Figure 3 shows the final geometries of the leading and trailing edges, respectively. As can be seen the resulting geometries are smooth.

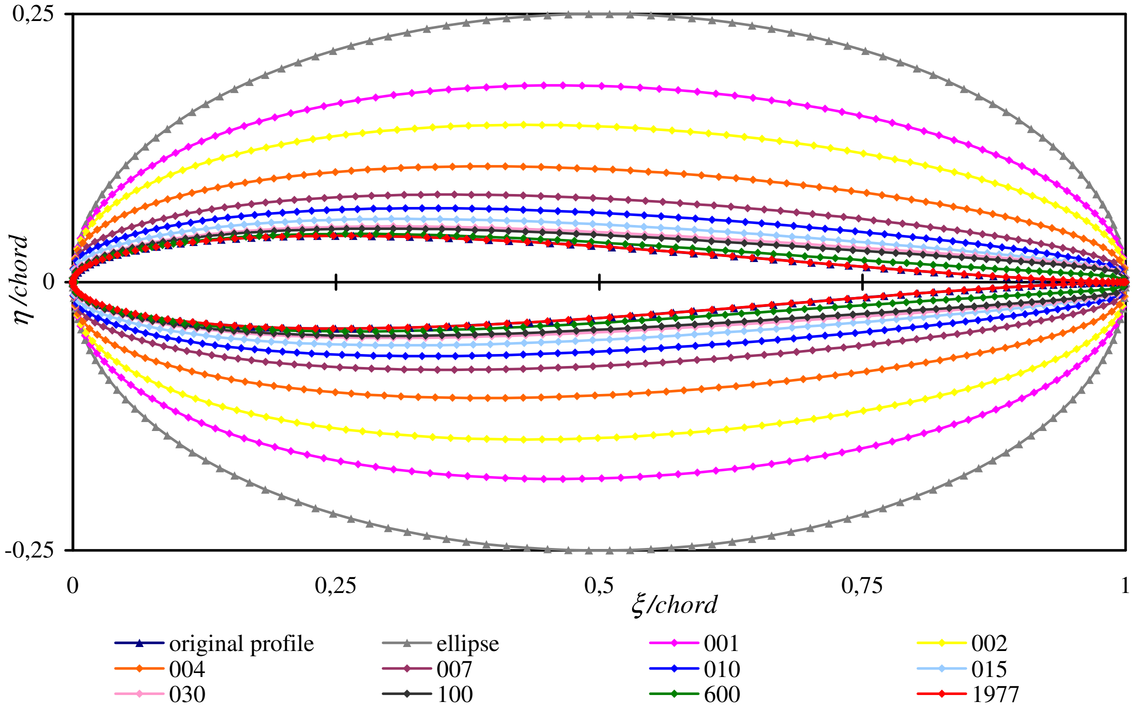

Figure 2. Evolution of the calculated foil shapes of a Joukowski symmetric airfoil ($2N = 160$)

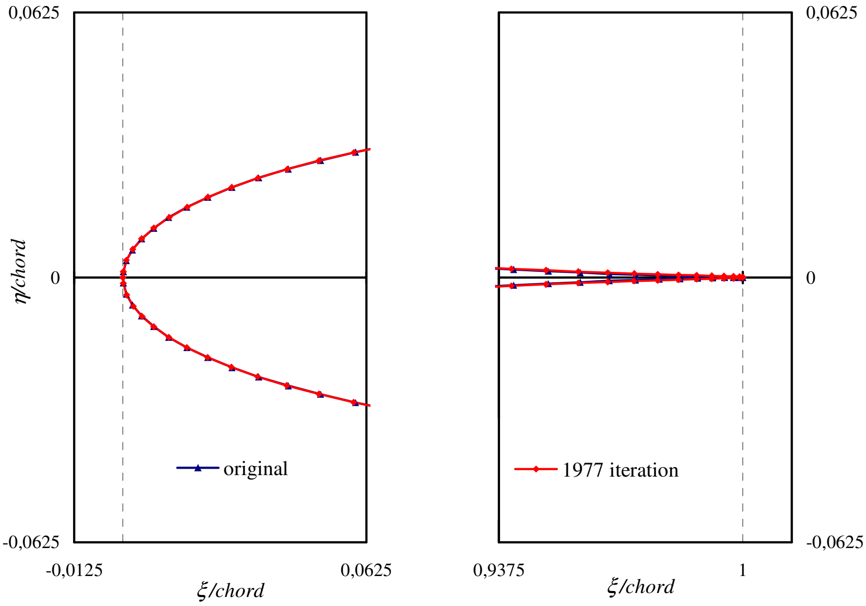

Figure 3. Final geometries of the leading and trailing edges of the symmetric Joukowski airfoil

4.2. Design of the symmetric Joukowski airfoil with angle of attack

In the study case of the Joukowski airfoil with angle of attack the ratio b/r_0 is taken as 0,9333 and the angle of attack is 10°. The details of this airfoil are shown in Fig. 4 which shows the original and final results predicted by the vortex distribution model and the original airfoil calculated by the Joukowski transformation.

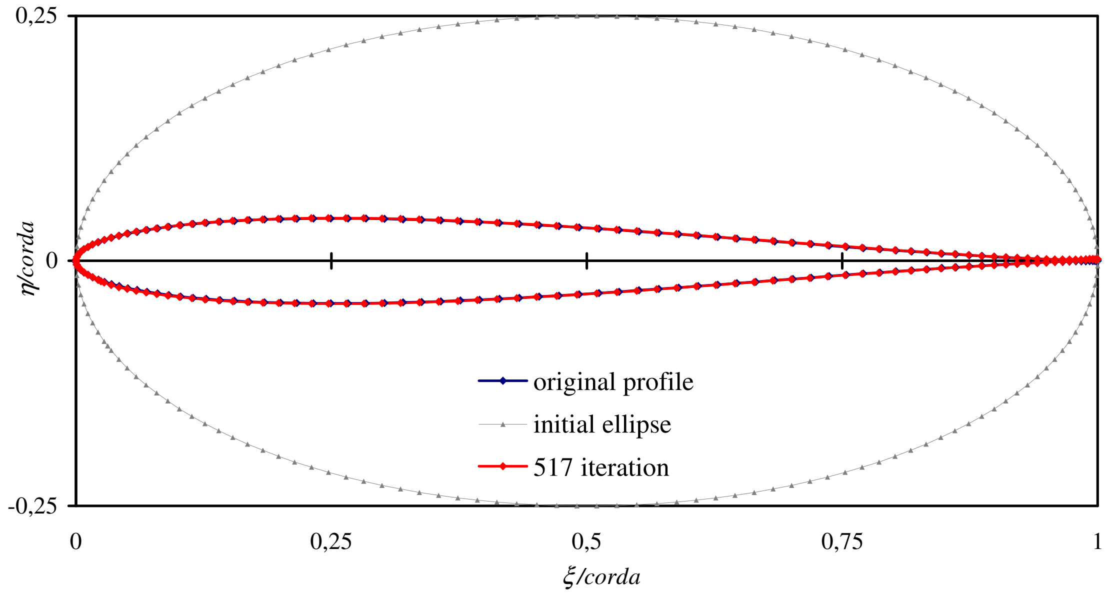

Figure 4. Design of a Joukowski symmetric airfoil ($2N = 160$)

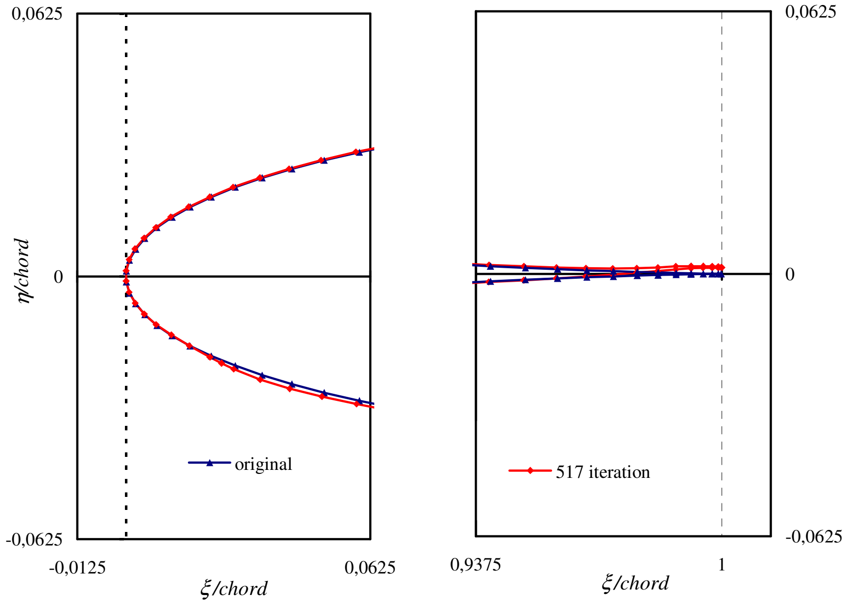

Figure 5. Final geometries of the leading and trailing edges of the symmetric Joukowski airfoil ($\beta_\infty = 10^\circ$)

Figure 5 shows the geometries of the leading and trailing edges of the original airfoil and the numerical prediction from the vortex distribution analysis. As can be seen the agreement is quite good, the difference between the original and final geometries of the airfoil is less than $0,2\%$ of the airfoil chord.

4.3. Design of the NACA 65-1210 airfoil

The NACA 65-1210 is used for the third study case. Its data was obtained from Abbott and Von Doenhoff (1959). Figure 6 shows the original airfoil NACA 65-1210 and the numerical predictions, showing good agreement, while Fig.7 shows the final numerical result of the leading and trailing edge geometries after 2000 iterations. Again the agreement between the original airfoil and the numerical predictions is quite good, as can be judged from the ratio of the thickness difference between the predicted value and the original one divided by chord length calculated at the trailing edge which is found to be less than $0,3\%$.

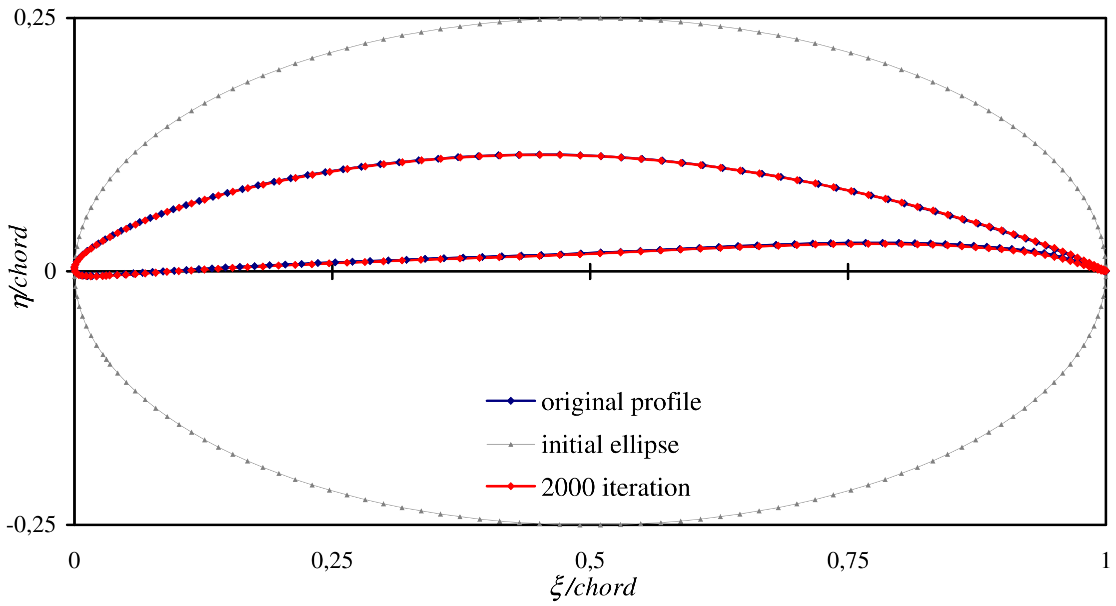

Figure 6. Design of the NACA 65-1210 airfoil ($2N = 160$)

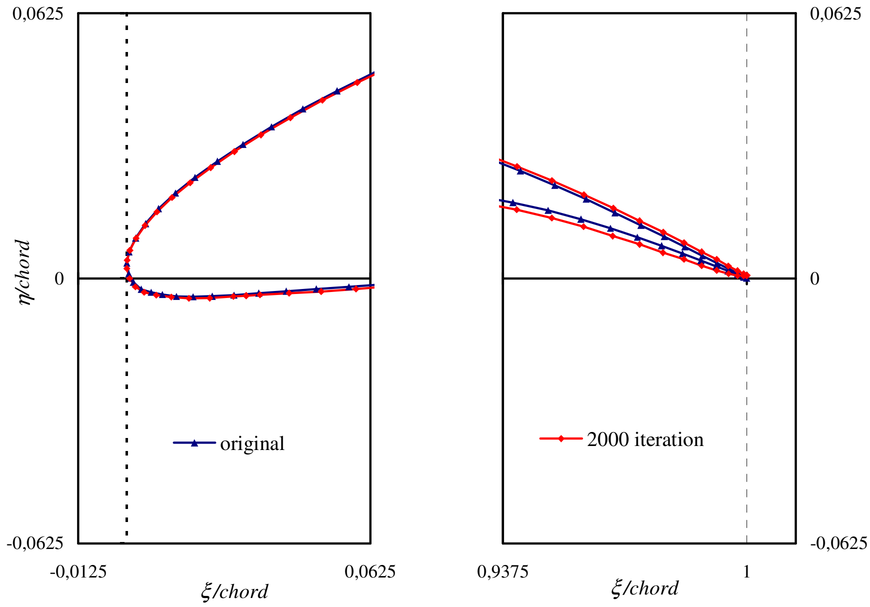

Figure 7. Final geometries of the leading and trailing edges of the non symmetric airfoil NACA 65-1210

4.4. Analysis of the NACA 65-1210 cascade

As an application of the computational code based upon the method of vortex distribution over the foil surface, it was used to design a NACA 65-1210 cascade. This cascade is used because there is available data to compare with the numerical predictions from the present code.

Figure 8 shows the numerical predictions from the present code as compared with experimental results and the Ackeret's method (Horlock, 1973). As can be seen the comparison is reasonably good.

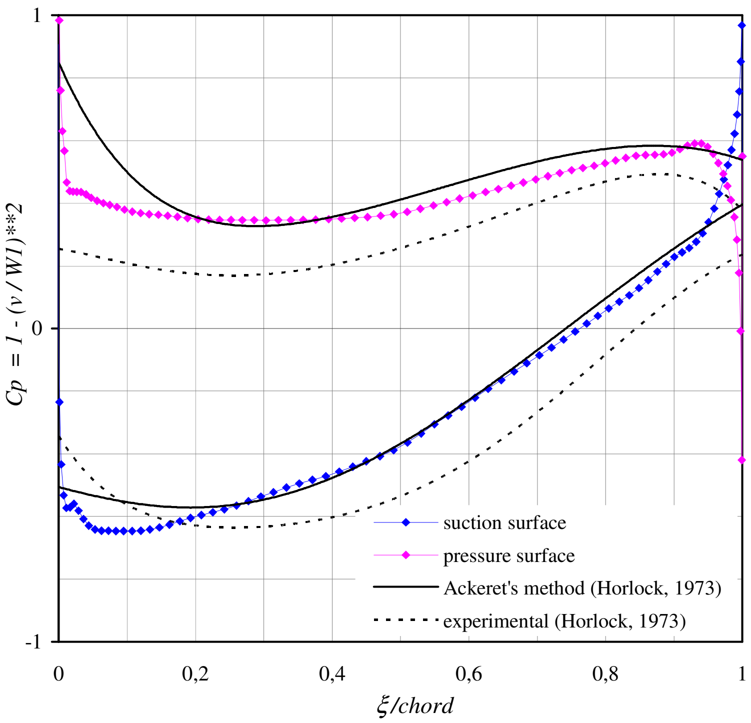

Figure 8. Pressure coefficient distribution over the surfaces of the airfoil NACA 65-1210 in cascade (stagger $\chi = 30^\circ$, pitch chord ratio $t/l = 1{,}0$) for $\beta_1 = 45^\circ$; the present code results $\beta_2 = 23{,}5^\circ$ with $2N = 160$ control points

5. CONCLUSIONS

The method of vortex distribution over the foil surface is used in the reverse sense, in this case initially a velocity distribution is specified over the foil and specially at the front stagnation point. The formulated model is set up into a numerical code which was extensively tested and optimized. The code was then used to design isolated airfoils, three of which are presented here, two analytical airfoils (Joukowski) and one experimental (NACA 65-1210). The Joukowski airfoils were calculated by using the Joukowski transformation while the NACA data was used for the NACA airfoil. The predicted results using the numerical code were compared with available results and good agreement was found. Finally the method and code were used to calculate a cascade and the results were compared with Horlock's results showing reasonably good agreement.

6. ACKNOWLEDGEMENTS

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