BAYESIAN ESTIMATION OF BOUNDARY HEAT FLUX IN A THIN PLATE

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Abstract. *In this paper we describe the solution of an inverse heat conduction problem dealing with the estimation of a boundary heat flux. Such heat flux is imposed on the surface of a thin metal plate and temperature measurements are considered to be taken over the non-heated surface. For the direct problem, a lumped formulation is used across the plate, so that the three-dimensional problem is formulated in two dimensions and in terms of an average transversal temperature. The inverse problem is solved with a Bayesian approach that is basically concerned with the analysis of the posterior probability density, which is the conditional probability of the parameters given the measurements. In the Bayesian approach to statistics, an attempt is made to utilize all available information in order to reduce the amount of uncertainty present in an inferential or decision-making problem. As new information is obtained, it is combined with any previous information to form the basis for statistical procedures. Simulated temperature measurements are used in the inverse analysis in order to show the capabilities of the proposed approach.*

Keywords*: Heat Conduction, Inverse Problem, Function Estimation, Bayesian Inference*

1. INTRODUCTION

In conventional heat conduction problems, the temperature distributions of a solid body are calculated when the boundary/initial conditions are known. Conversely, inverse problems involve the estimation of the cause from the knowledge of the effect. For example, an inverse heat conduction problem (IHCP) may deal with the estimation of an unknown boundary heat flux by utilizing transient temperature measurements in the solid.

From the mathematical point of view, inverse problems are unstable and belong to the class of ill-posed problems (A.N. Tikhonov and V. Y. Arsenin 1977, J. Hadarmard 1923, J. Kaipio 2005, O. M. Alifanov 1994, Ozisik and Orlande 2000). In order to overcome such instabilities, there are generally two main frameworks for the solution of ill-posed inverse problems: the traditional regularization methods (A.N. Tikhonov and V. Y. Arsenin 1977, H. Engl 1992) and the statistical inversion methods (J. Kaipio 2005).

The statistical inversion approach is based on the Bayesian statistical framework in which (probability distribution) models for the measurements and the unknowns are constructed separately and explicitly. The solution of the inverse problem is recast in the form of statistical inference from the posterior probability density, which is our model for the conditional probability distribution of the unknown parameters given the measurements. The measurement model incorporating the measurement error model and the related uncertainties is called the likelihood, that is, the conditional probability of the measurements given the unknown parameters. The model for the unknowns that reflects all the uncertainty of the parameters without the information conveyed by the measurements is called the prior model (J. Kaipio, 2005).

Despite the fact that the minimization of the least-squares norm is indiscriminately used, it is a non-Bayesian estimator. A Bayesian estimator (J. Kaipio, 2005) is basically concerned with the analysis of the posterior probability density, which is the conditional probability of the parameters given the measurements, while the likelihood is the conditional probability of the measurements given the parameters. If we assume the parameters and the measurement errors to be independent Gaussian random variables, with known means and covariance matrices, and that the measurement errors are additive, a closed form expression can be derived for the posterior probability density. In this case, the estimator that maximizes the posterior probability density can be recast in the form of a minimization problem involving the maximum a posteriori objective function. On the other hand, if different prior probability densities are assumed for the parameters, the Posterior Probability Distribution does not allow an analytical treatment. In this case, Markov Chain Monte Carlo (MCMC) methods are used to draw samples of all possible parameters, so that inference on the posterior probability becomes inference on the samples.

In this paper a time-dependent heat flux is estimated by employing a Markov Chain Monte Carlo (MCMC) through the implementation of the Metropolis-Hastings algorithm (Gamerman and Lopes 2006, Lee 2004, Migon and Gamerman 1999, Orlande *et al*. 2008). The physical problem of interest consists of a horizontal thin plate heated with a circular electrical resistance. For the formulation of the physical problem, a three-dimensional transient formulation is first reduced into a two-dimensional problem by using a lumped formulation along the thickness of the plate. The numerical solution for the direct problem was implemented in the *C++* platform, based on the finite element method (FEM) with a triangular elements discretization. Simulated experimental data for the temperature distribution in different positions and time are used in the inverse analysis, in order to show the capabilities of the proposed approach.

2. DIRECT PROBLEM

The physical problem under analysis is graphically presented by Figs. (1a,b). An horizontal plate with thickness *Lz* and lateral dimensions *Lx* x *Ly* is assumed to be heated by a circular electrical resistance, installed in the center of the back surface, which provides a time dependent heat flux. The lateral surfaces were considered to be insulated, while the top boundary looses heat by convection and radiation to its surroundings. In other to further challenge the present heat flux inversion analysis, the simulated experimental temperature was supposed to be taken on the non-heated surface with an infrared camera.

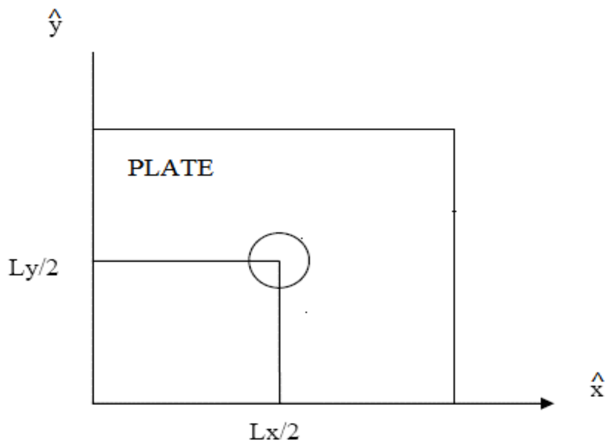

Figure 1.a – Top view of the heated plate

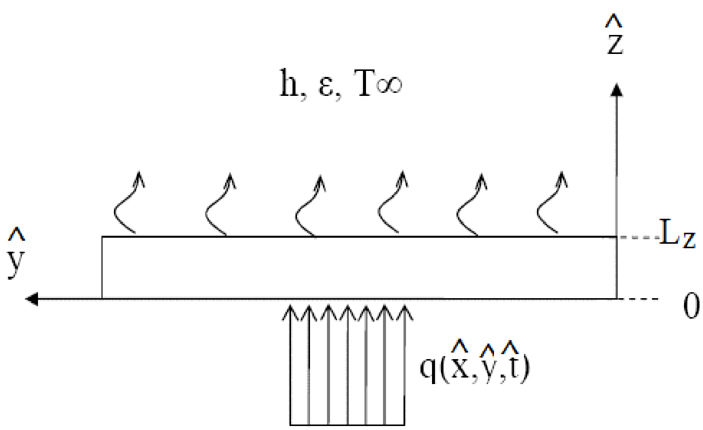

Figure 1.b – Lateral view of the heated thin plate.

The mathematical model which describes the temperature distribution on the plate is given by:

$$\frac{1}{\hat{\alpha}}\frac{\partial T}{\partial \hat{t}} = \frac{\partial^2 T}{\partial \hat{x}^2} + \frac{\partial^2 T}{\partial \hat{y}^2} + \frac{\partial^2 T}{\partial \hat{z}^2} \quad \text{at} \quad 0 < \hat{x} < L_x, \quad 0 < \hat{y} < L_y, \quad 0 < \hat{z} < L_z, \quad \hat{t} > 0 \tag{1.a}$$

with boundary and initial conditions, given by

$$\frac{\partial T}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = 0, \quad \hat{t} > 0 \tag{1.b}$$

$$\frac{\partial T}{\partial \hat{x}} = 0 \quad \text{at} \quad \hat{x} = L_x, \quad \hat{t} > 0 \tag{1.c}$$

$$\frac{\partial T}{\partial \hat{y}} = 0 \quad \text{at} \quad \hat{y} = 0, \quad \hat{t} > 0 \tag{1.d}$$

$$\frac{\partial T}{\partial \hat{y}} = 0 \quad \text{at} \quad \hat{y} = L_y, \quad \hat{t} > 0 \tag{1.e}$$

$$-\hat{k}\frac{\partial T}{\partial \hat{z}} = q(\hat{x}, \hat{y}, \hat{t}) \quad \text{at} \quad \hat{z} = 0, \quad \hat{t} > 0 \tag{1.f}$$

$$-\hat{k}\frac{\partial T}{\partial \hat{z}} = \hat{h}(T - T_\infty) + \varepsilon\sigma(T^4 - T_\infty{}^4) \quad \text{at} \quad \hat{z} = L_z, \quad \hat{t} > 0 \tag{1.g}$$

$$T(\hat{x}, \hat{y}, \hat{z}, 0) = T_0 \quad \text{at} \quad \hat{t} = 0,\ 0 < \hat{x} < L_x, \quad 0 < \hat{y} < L_y, \quad 0 < \hat{z} < L_z \tag{1.h}$$

Since we are dealing with a very thin plate made of a good heat conductor, the above heat conduction problem can be rewritten by neglecting temperature gradients along the $\hat{z}$ direction and it can be written in terms of the average temperature $\overline{T}(\hat{x}, \hat{y}, \hat{t})$ defined as

$$\overline{T}(\hat{x}, \hat{y}, \hat{t}) = \frac{1}{L_z}\int_{z'=0}^{z'=L_z} T(\hat{x}, \hat{y}, \hat{z}, \hat{t})dz' \tag{2}$$

The formulation for the problem based on the partial lumping approach described above is given in dimensionless form as:

$$\frac{1}{\alpha}\frac{\partial \theta}{\partial t} = \frac{\partial^2 \theta}{\partial x^2} + \frac{\partial^2 \theta}{\partial y^2} - h(\theta)\theta + g(x, y, t) \quad \text{at} \quad 0 < x < 1, \quad 0 < y < 1, \quad t > 0 \tag{3.a}$$

$$\frac{\partial \theta}{\partial x} = 0 \quad \text{at} \quad x = 0, \quad t > 0 \tag{3.b}$$

$$\frac{\partial \theta}{\partial x} = 0 \quad \text{at} \quad x = 1, \quad t > 0 \tag{3.c}$$

$$\frac{\partial \theta}{\partial y} = 0 \quad \text{at} \quad y = 0, \quad t > 0 \tag{3.d}$$

$$\frac{\partial \theta}{\partial y} = 0 \quad \text{at} \quad y = 1, \quad t > 0 \tag{3.e}$$

$$\theta(x, y, 0) = 0 \quad \text{at} \quad t = 0 \quad 0 < x < 1, \quad 0 < y < 1 \tag{3.f}$$

where $h(\theta)$ is the linearized source term that takes into account the contributions of the boundary conditions at $\hat{z} = 0$, and $\hat{z} = L_z$, that is,

$$h(\theta) = \frac{L^2}{L_z k_R}[\hat{h} + 4\varepsilon\sigma(\theta q_R \frac{L}{k_R} + T_\infty)^3] \tag{3.g}$$

$$g(x, y, t) = \frac{L}{L_z}\phi(x, y, t) \tag{3.h}$$

By assuming $L=L_x=L_y$ the following dimensionless groups were defined to obtain problem (3.a-h):

$$\alpha = \frac{\hat{\alpha}}{\alpha_R}, \quad t = \frac{\hat{t}\alpha_R}{L^2}, \quad x = \frac{\hat{x}}{L}, \quad y = \frac{\hat{y}}{L}, \quad \theta = \frac{\overline{T} - T_\infty}{q_R\, L / k_R}, \quad \phi(x, y, t) = \frac{q(\hat{x}, \hat{y}, \hat{t})}{q_R(\hat{x}, \hat{y}, \hat{t})} \tag{4.a-e}$$

where the subscripts R in Eqs. (4.a-e) refer to reference values of the physical variables.

The numerical solution of the direct problem given by Eqs. (3.a-h) is based on the finite element method. In this work triangular elements were used for the plate discretization as presented by Fig. 2.

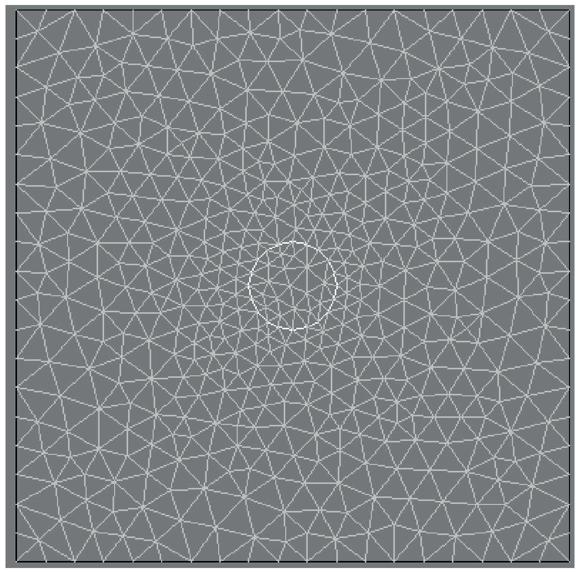

Figure 2 - Mesh of plate surface into triangular elements used to the code implementation based on FEM.

3. INVERSE PROBLEM

The inverse problem of interest for this work involves the estimation of the imposed boundary heat flux $\phi(x, y, t)$. In this work such flux was assumed to have a known spatial distribution given by the location of the electrical resistance. Such a location is illustrated in Fig. 2. Therefore, the present inverse analysis deals only with the estimation of its time variation, represented heretofore as $\phi(t)$.

For the solution of the inverse problem, the unknown function $\phi(t)$ was parameterized in the form:

$$\phi(t) = \sum_{i=1}^{I} \phi_i U_i(t) \tag{5.a}$$

where $U_i(t)$ is the unit step function, that is,

$$U_i(t) = \begin{cases} 1 & , \quad for\ t_i < t < t_{i+1} \\ 0 & , \quad elsewhere \end{cases} \tag{5.b}$$

In the Bayesian approach to statistics, an attempt is made to utilize all available information in order to reduce the amount of uncertainty present in an inferential or decision making problem. As new information is obtained, it is combined with any previous information to form the basis for statistical procedures. The formal mechanism used to combine the new information with the previously available information is known as Bayes' theorem (R. Winkler, 2003). Therefore, the term Bayesian is often used to describe the so-called statistical inversion approach, which is based on the following principles (Migon and Gamerman, 1999):

1. All variables included in the model are modeled as random variables.
2. The randomness describes the degree of information concerning their realizations.

3. The degree of information concerning these values is coded in probability distributions.
4. The solution of the inverse problem is the posterior probability distribution.

The vector of parameters to be estimated is given by:

$$\mathbf{P}^{\mathrm{T}} = (\phi_1, \phi_2, \dots, \phi_I) \tag{6}$$

and the vector of available measurements as

$$\mathbf{Y}^{\mathrm{T}} = (Y_1, Y_2, \dots Y_m, \dots, Y_M) \tag{7}$$

where I is the number of parameters and M is the number of measurements. Bayes' theorem can then be stated as (J. Kaipio, 2005):

$$\pi_{\text{posterior}}(\mathbf{P}) = \pi(\mathbf{P}|\mathbf{Y}) = \frac{\pi(\mathbf{Y}|\mathbf{P})\pi_{\text{prior}}(\mathbf{P})}{\pi(\mathbf{Y})} \tag{8}$$

where $\pi_{\text{posterior}}(\mathbf{P})$ is the posterior probability density, that is, the conditional probability of the parameters **P** given the measurements **Y**; $\pi_{\text{prior}}(\mathbf{P})$ is the prior density, that is, the coded information about the parameters prior to the measurements; $\pi(\mathbf{Y}|\mathbf{P})$ is the likelihood function, which expresses the likelihood of different measurement outcomes **Y** with **P** given; and $\pi(\mathbf{Y})$ is the marginal probability density of the measurements, which plays the role of a normalizing constant. In practice, such normalizing constant is difficult to compute and numerical techniques, like Markov Chain Monte Carlo Methods, are required in order to obtain samples that represent accurately the posterior probability density. The numerical method most used to explore the space of states of the posteriori distribution is the Monte Carlo simulation. The Monte Carlo simulation is based on a large sample of the probability density function (in this case, the function of the posterior probability density $\pi(\mathbf{P}|\mathbf{Y})$).

An important practical question is how the initial values influence the behavior of the chain. The idea is that as the number of iterations increases, the chain gradually converges to an equilibrium distribution. Thus, generally the initial states are discarded until the chain reaches equilibrium. The problem then is to build algorithms that generate the Markov chain whose distribution converges to the distribution of interest. One of the most commonly used MCMC method is the Metropolis-Hastings algorithm (Gamerman and Lopes 2006, Migon and Gamerman 1999).

In order to implement the MCMC, a density $q(\mathbf{P}^*,\mathbf{P}^{(t-1)})$ is required, which gives the probability of moving from the current state in the chain $\mathbf{P}^{(t-1)}$ to a new state $\mathbf{P}^*$. The Metropolis-Hastings algorithm (Gamerman and Lopes 2006, J. Kaipio 2005, Migon and Gamerman 1999) used in this work to implement the MCMC method can be summarized in the following steps:

1.Sample a candidate point $\mathbf{P}^*$ from a jumping distribution $q(\mathbf{P}^*,\mathbf{P}^{(t-1)})$.

2. Calculate:

$$\alpha = \min\left[1, \frac{\pi(\mathbf{P}^*|\mathbf{Y})q(\mathbf{P}^{(t-1)}|\mathbf{P}^*)}{p(\mathbf{P}^{(t-1)}|\mathbf{Y})q(\mathbf{P}^*|\mathbf{P}^{(t-1)})}\right] \tag{9}$$

3. Generate a random value U which is uniformly distributed on (0,1).

4. If $U \le \alpha$ define $\mathbf{P}^{(t)} = \mathbf{P}^*$; otherwise, define $\mathbf{P}^{(t)} = \mathbf{P}^{(t-1)}$.
5. Return to step 1 in order to generate the sequence $\{\mathbf{P}^{(1)}, \mathbf{P}^{(2)}, \dots, \mathbf{P}^{(n)}\}$.

The success of the method depends the acceptance rate and on proposals that are easy to simulate. The method replaces a difficult to generate $\pi(\mathbf{P}|\mathbf{Y})$ by several generations of the proposal q.

In this study we have chosen to adopt symmetrical chains, and for the Metropolis-Hastings algorithm the notion of symmetric chain is applied to the proposed transition q. Thus, q defines a single transition around the earlier position in the chain, i.e., $q(\mathbf{P}^*,\mathbf{P}^{(t-1)}) = q(\mathbf{P}^{(t-1)},\mathbf{P}^*)$ for all $(\mathbf{P}^*, \mathbf{P}^{(t-1)})$. In this case, Eq. (9) does not depend on q. For more details on theoretical aspects of the Metropolis-Hastings algorithm and MCMC methods, the reader should consult references (Gamerman and Lopes 2006, J. Kaipio 2005, Migon and Gamerman 1999).

We assume in this work that the measurement errors were additive, uncorrelated, normally distributed, with zero mean and a constant standard deviation and independent of the unknown parameters. The simulated measurements were then generated with Eq. (10), where ω is a random number with normal distribution and unitary standard deviation, and $\sigma = 0.01\ T^{\max}$, that is 1% of the maximum exact temperature obtained from the solution of the direct problem

$$Y_m = T_m^{\text{exact}} + \omega\sigma \tag{10}$$

Hence, the likelihood function is given by Eq. (11) (Gamerman and Lopes 2006, J. Kaipio 2005, J. V. Beck and K. Arnold 2005, Migon and Gamerman 1999):

$$\pi(\mathbf{Y}|\mathbf{P}) = (2\pi)^{-1/2}\left|\mathbf{W}^{-1}\right|^{-1/2}\exp[-\frac{1}{2}\left(\mathbf{Y}-\mathbf{T}(\mathbf{P})\right)^T \mathbf{W}\left(\mathbf{Y}-\mathbf{T}(\mathbf{P})\right)] \tag{11}$$

where $\boldsymbol{T}$ is the estimated temperature, obtained from the solution of the direct problem with estimates for the vector of unknown parameters **P**, **Y** is the vector of measured temperatures and **W** is the inverse of the covariance matrix of the measurements.

4. RESULTS AND DISCUSSIONS

The inverse problem solution here illustrated involves the analysis of a step variation in time of the heat flux function, as presented in Fig 3. In order to examine the accuracy and robustness of the proposed inverse analysis approach, we made use of simulated transient measured temperature data such as obtained through infrared thermography. Therefore, measurements are supposed to be available for each of the finite elements used in the discretization of the direct problem. For the results of the inverse analysis to be presented below, we have employed the parameter values shown in Table 1. With such parameters, a typical transient measurement for one of the camera pixels (coincident with a finite element in the discretization), is illustrated in Fig. 3.b.

For the solution of the inverse problem, the prior information for the parameters was considered in the form of a uniform distribution.

Table 1 – Parameter employed in simulated experimental data

Plate Material	Brass $\hat{k} = 111\ W/(mK)$ $\hat{\alpha} = 3.41\text{x}10^{-5} m^2 s$ $\varepsilon = 0.97$
Imposed Heat Flux in the Electrical Resistance	$q(\hat{x},\hat{y},\hat{t}) = q_0 = 18255\ W/(m^2K)$ for $30s < \hat{t} < 90s$ $q(\hat{x},\hat{y},\hat{t}) = 0$ elsewhere
Reference Values	$k_R = \hat{k}$ $\alpha_R = \hat{\alpha}$ $q_R(\hat{x},\hat{y},\hat{t}) = q_0$
Electrical Resistance Diameter	$25.4mm$
Plate Dimensions	Lx = 160 mm; Ly = 160 mm; Lz = 1 mm
Number of Experimental Measurements in time	M = 120
Number of Parameter to be estimated	I = 120

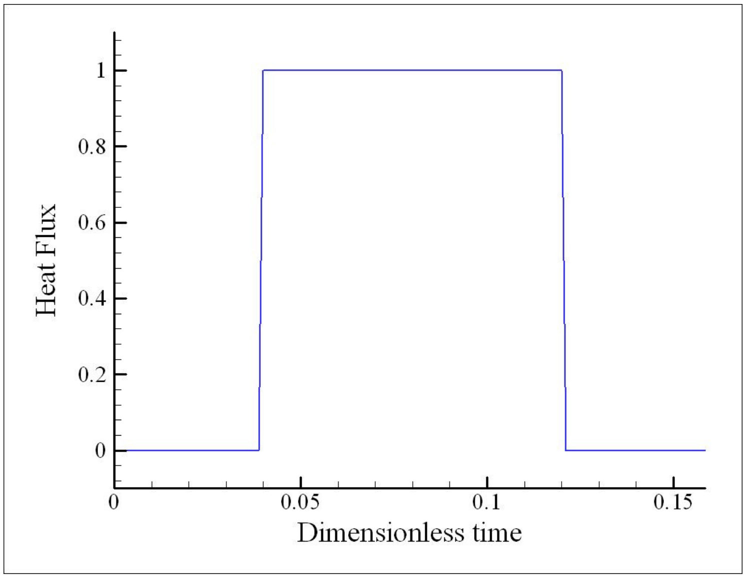

Figure 3.a – Exact heat flux imposed in the simulated experimental data in dimensionless form

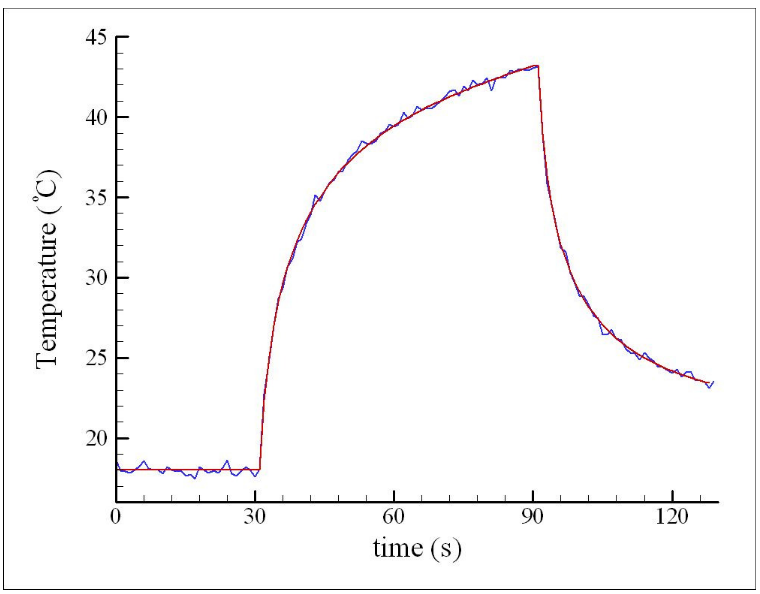

Figure 3.b – Exact Temperature (red curve) and Experimental Temperature as presented in Eq. 10 (blue curve)

Figures 4.a,b illustrate the Markov chain evolutions along a total of 50,000 states for 5 different parameters, $P_1 = \phi(t = 0.0265625)$, $P_2 = \phi(t = 0.053125)$, $P_3 = \phi(t = 0.0796875)$, $P_4 = \phi(t = 0.10625)$ and $P_5 = \phi(t = 0.132813)$, (red curve, green curve, blue curve, magenta curve and cyan curve, respectively). In Fig 4.a the candidate parameters were generated with a search step in the Metropolis-Hastings algorithm of 0.001, while in Fig. 4.b this step was of 0.005. These figures show that both values of the search step are suitable to generate the samples. In fact, for both cases the samples do not exhibit low-frequency oscillations and generally reach equilibrium within the number of states used for the simulation.

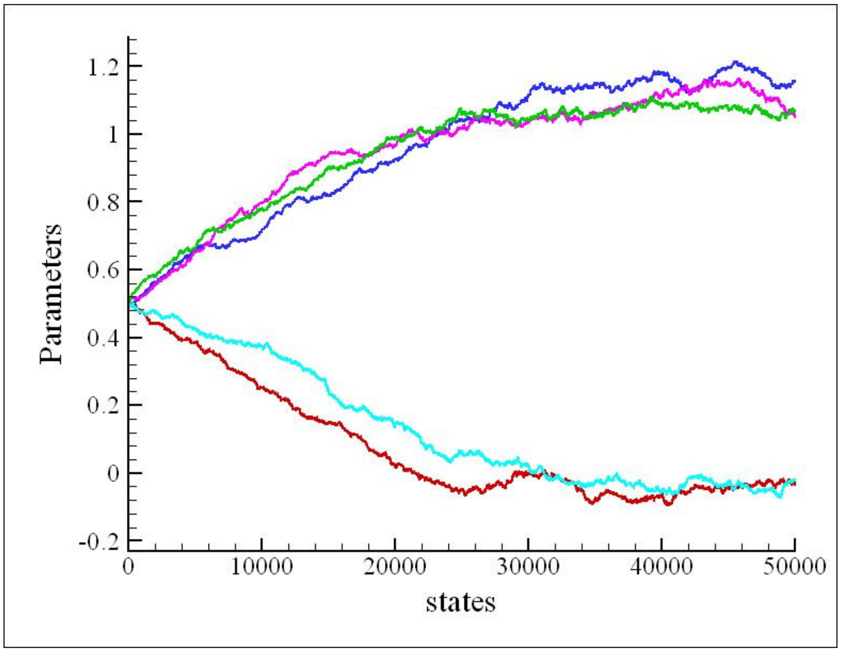

Figure 4.a - Markov chain evolutions for search step of 0.001, for 5 different parameters

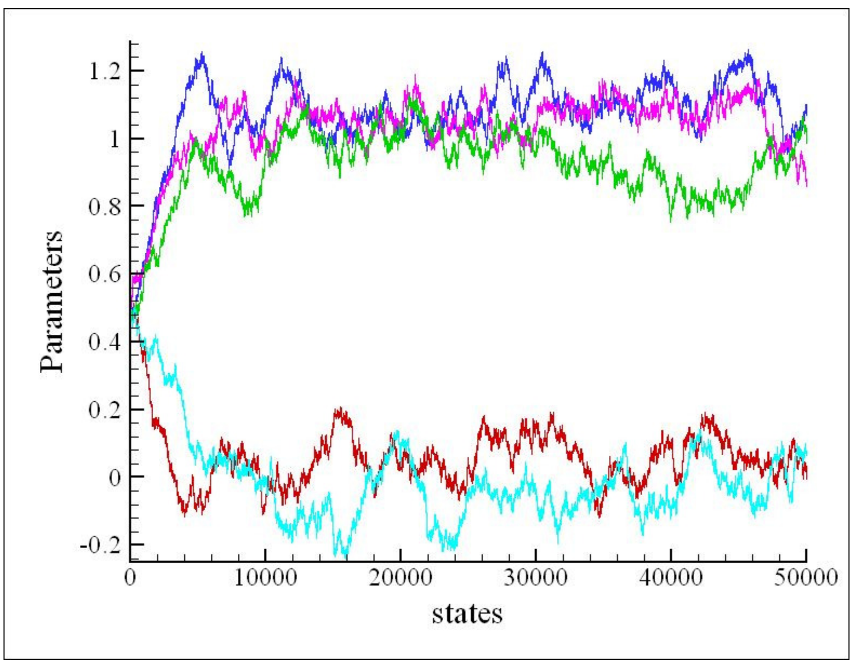

Figure 4.b - Markov chain evolutions for search step of 0.005, for 5 different parameters

Figures 4.a,b show that the burn-in periods, where the samples in the Markov chains are discarded, should be of 25,000 and 10,000 states, for search steps of 0.001 and 0.005, respectively. As expected, the burn-in period is longer for a smaller search step, because the sample space takes longer to be explored. The mean values for five different parameters, corresponding to the heat fluxes at five different times, are shown in Tables 2.a,b, for search steps of 0.001 and 0.005, respectively. The limits of the 95% confidence intervals for such parameters are also illustrated in these tables. These tables show that the limited number of states used for the computation of the statistics of each parameter with the step of 0.001 give 95% confidence intervals that do not include some of the exact parameters. On the other hand, an excellent agreement between the exact parameters and their corresponding estimated means, is obtained with a search step of 0.005.

Table 2.a –Estimations for 5 different parameters for a search step of 0.001

Parameter	Exact	MCMC	Min. 95%	Max. 95%
t = 0.026562	0	-0.043450	-0.086507	0.001991
t = 0.053125	1	1.138400	1.04407	1.203980
t = 0.079687	1	1.084660	1.02429	1.154740
t = 0.106250	1	1.067990	1.04124	1.095040
t = 0.132812	0	0.104279	0.049261	0. 143583

Table 2.b – Estimations for 5 different parameters for a search step of 0.005

Parameter	Exact	MCMC	Min. 95%	Max. 95%
t = 0.026562	0	0.055909	-0.058429	0.175347
t = 0.053125	1	1.100760	0.991847	1.226630
t = 0.079687	1	1.061470	0.945201	1.151260
t = 0.106250	1	0.951897	0.801431	1.086190
t = 0.132812	0	-0.049778	-0.200665	0.107743

Figure 5 presents the heat fluxes estimated with search steps of 0.001 and 0.005. In this figure, the black line gives the exact heat flux, while the red and blue lines present the estimated fluxes with the search steps of 0.001 and 0.005, respectively. The analysis of this figure reveals a quite good agreement between exact and estimated heat fluxes, despite the use of a non-informative prior given by a uniform distribution. Also, this figure shows reasonably stable estimates without the use of a smoothness prior given in the form of a Markov random field (J. Kaipio 2005).

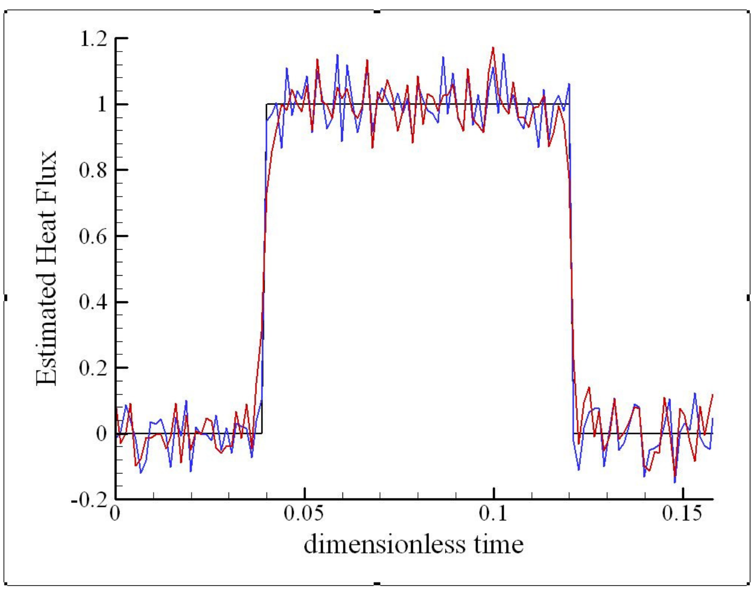

Figure 5 – Comparison between the exact and the estimated heat flux, with 120 parameters for a search step of 0.001 (red curve) and 0.005 (blue curve)

5. CONCLUSIONS

In this work we applied a Bayesian approach to the estimation of a boundary heat imposed on a thin metallic plate. The physical problem is formulated in terms of partial lumping along the thickness of the plate. The spatial distribution of the heat flux is supposed to be known in advance, so that only its transient variation is estimated with the inverse analysis. The Metropolis-Hastings algorithm of the Markov Chain Monte Carlo method is used in this work.

The results obtained in this paper for a strict test-case, involving the estimation of a discontinuous function, reveals the accuracy and robustness of the present approach for the solution of the inverse problem. In fact, quite accurate results were obtained even a non-informative prior given by a uniform distribution, without the use of a smoothness prior.

6. ACKNOWLEDGEMENTS

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7. REFERENCES

A.N. Tikhonov, V.Y. Arsenin, Solution of ill-posed problems, Winston & Sons, Washington, DC, 1977.

Beck, J. and Arnold, K., (1977), Parameter Estimation in Engineering and Science, Wiley Interscience, New York.

Gamerman, D., and Lopes, H.F. (2006) *Markov Chain Monte Carlo: Stochastic Simulation for Bayesian Inference*, Chapman & Hall/CRC, 2nd edition, Boca Raton, FL.

H. Engl, Inverse Problems, Johannes–Kepler–Universität, Linz, Austria, 1992, Lecture notes.

J. Hadamard, Lectures on Cauchy's Problem in Linear Differential Equations, Yale University Press, New Haven, CT, 1923.

J. Kaipio, E. Somersalo, Statistical and Computational Inverse Problems, Springer, New York, 2005.

J.V. Beck and K. Arnold, *Parameter Estimation in Engineering and Science*, Wiley Interscience, New York (1977).

Lee, P. (2004), *Bayesian Stat*istics, Oxford University Press, London.

Migon, H.S., and Gamerman, D. (1999),"Statistical Inference: An Integrated Approach", Arnold/Oxford, London/New York.

O.M. Alifanov, Inverse Heat Transfer Problems, Springer-Verlag, New York, 1994.

Orlande, H.R.B., Colaço, M. & Dulikravich, G.S. (2008), "Approximation of the likelihood function in the Bayesian techinique for the solution of inverse problems", Inverse Problems in Science and Engineering, v.16, pp. 677-692.

Ozisik, M.N. and Orlande, H.R.B. (2000) *Inverse Heat Transfer: Fundamentals and Applications*, Taylor and Francis, New York.

R. Winkler, *An Introduction to Bayesian Inference and Decision*, Probabilistic Publishing, Gainsville, (2003).