CONTRIBUTION TO THE ESTIMATION OF EQUIVALENT SHEAR STRESS AMPLITUDE

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Abstract. *A new definition is proposed for assessing the equivalent shear stress amplitude. This method generalised to the three-dimensional complex loadings the minimum ellipse method in a 5 dimensional space. Sinusoidal loadings examples are presented in order to illustrated the definition.*

Keywords: *amplitude of the second invariant of the stress deviator, non-proportional and asynchronous loading, prismatic hull*

1. INTRODUCTION

The estimation of the amplitude of the second invariant of the stress deviator (or equivalent shear stress) $\sqrt{J_{2,a}}$ is a recurrent problem when dealing with multiaxial fatigue. This problem is particularly important for evaluating the fatigue behaviour of mechanical structures, frequently submitted to complex random loads.

Many methods have been developed in recent years to get a good measure of this amplitude. In order to simplify the calculation, a change of variables is commonly introduced on the stress deviator to consider in a five dimensions Euclidean space E_5. In this space, the amplitude can be defined by various methods such as the smallest circumscribed hypersphere (Papadopoulos *et al.*, 1997, Bernasconi, 2002) or the minimum circumscribed ellipse (Li and De Freitas, 2002, Cristofori *et al.*, 2008). Recently, the Mamiya*'s research team* have introduced the concept of the smallest ellipsoid (Gonçalves *et al.*, 2004; Gonçalves *et al.*, 2005; Zouain *et al.*, 2006; Balthazar *et al.*, 2007) and of the prismatic hull to generalize the method proposed in Li and De Freitas (2002).

The method proposed in this work fits into this logic. It generalizes the method proposed in the work of Li and De Freitas (2002) by investigating the principal axes of the load path to obtain the prismatic hull circumscribed to this path. The originality of the method is that we obtain the semi-axis of this hull by a search of the principal directions in E_5.

The first part presents a state of the art methods from the literature cited above. In a second part we develop our definition, then, in a third party, we illustrate the method with examples involving sinusoidal loads.

2. STATE OF THE ART OF DEFINITIONS OF $\sqrt{J_{2,a}}$ ESTIMATED IN E_5

Many authors, as for example Sines or Crossland, consider $\sqrt{J_{2,a}}$ as a major parameter to control the initiation of cracks by assuming proportional loading paths, the octahedral plane being the plane of fatigue crack initiation (or maximum shear plane). In recent years, several methods have been proposed to give a definition of $\sqrt{J_{2,a}}$ suitable for more complex loadings. For example, the method of the minimum circumscribed circle, introduced by Dang Van and Papadopoulos (Papadopoulos *et al.*, 1997), allows to overcome the problem of non-uniqueness inherent in the method of the longest chord by the introduction of a single circle circumscribing the loading path Ψ.

However, to facilitate the calculation of $\sqrt{J_{2,a}}$, it is convenient to define a change of variable over the stress deviator considering in a 5-dimensions space E_5 (Lambert, 2007):

$$\begin{aligned} &S_1(t) = \frac{\sqrt{3}}{2}\left(\frac{2}{3}\sigma_{xx}(t) - \frac{1}{3}\sigma_{yy}(t) - \frac{1}{3}\sigma_{zz}(t)\right), \quad S_2(t) = \frac{1}{2}\sigma_{yy}(t) - \frac{1}{2}\sigma_{zz}(t), \\ &S_3(t) = \sigma_{xy}(t), \quad S_4(t) = \sigma_{xz}(t), \quad S_5(t) = \sigma_{yz}(t) \end{aligned} \tag{1}$$

where $\sigma_\bullet(t)$ stand for the stress tensor components. The amplitude $\sqrt{J_{2,a}}$ can be obtained in that space by calculating the minimum hypersphere circumscribed to the loading path (Papadopoulos *et al.*, 1997). For a periodic load, the tip of the vector $\mathbf{S}(t)$ then describes in E_5 a closed curve Φ. The length of the vector $\mathbf{S}_m$ (containing the mean of each component) that points to the center of this hypersphere, is obtained by $\mathbf{S}_m \ : \ \min_{\mathbf{S}'}\left(\max_{t\in T}\left\|\mathbf{S}(t) - \mathbf{S}'\right\|\right)$, where:

$$\sqrt{J_{2,a}} = \max_{t\in T}\left\|\mathbf{S}(t) - \mathbf{S}_m\right\| \tag{2}$$

It is easy to see that the proportional and non-proportional loading paths are circumscribed by the same sphere. This definition is therefore questionable, especially as the experiments carried out by some authors as Kueppers *et al.* (2006) showing that for ductile materials, the discrepancy between the solicitations of non-proportional loading induces a reduction in fatigue resistance.

It therefore seems necessary to take into account the evolution of the principal directions of the stress at the time for assessment of $\sqrt{J_{2,a}}$. Thus Deperrois (1991) proposes the following definition of $\sqrt{J_{2,a}}$:

$$\sqrt{J_{2,a}} = \frac{1}{2\sqrt{2}}\sqrt{\sum_{i=1}^{5} a_i^2} \tag{3}$$

In this expression, the components a_i are evaluated by determining first the longest chord a_5 between two distinct points in the load. This procedure is repeated for the following a_i searching again the longest chord in a reduced and orthogonal sub space to the sub-space generated by the previous chord.

However, the definition (3) suffers from non-uniqueness of the longest chord in some cases (Papadopoulos *et al.*, 1997). Alternatively Li and De Freitas (2002) then formalize the method of the minimum circumscribed ellipse. In this case, $\sqrt{J_{2,a}}$ is obtained through the semi-axes of this method as:

$$\sqrt{J_{2,a}} = \sqrt{R_1^2 + R_2^2} \tag{4}$$

where R_1 and R_2 are the minimum and maximum semi-axis of the ellipse formed by a load path in a plane. This method offers a better definition that the minimum circumscribed circle method as it allows to take into account the effects of phase shift. But this method only deals with biaxial loadings.

Gonçalves *et al.* (2004) then generalize the method proposed by Li and De Freitas (2002), suggesting to define the path of loading in a jacket elliptical minimum standard and as:

$$\sqrt{J_{2,a}} = \sqrt{\sum_{i=1}^{5} a_i^2} \tag{5}$$

In this expression, a_i become semi-axes of the ellipsoid circumscribing the loading path in E_5. The authors show that for multiaxial loading, synchronous and out-of-phases, the a_i correspond to the maximum amplitudes of the components $S_{i,a}$ of the stress deviator in E_5.

They also propose a variant of this extension with the notion of prismatic hull (Gonçalves *et al.*, 2005). In this case, the rectangular hull tangent to the loading, defines the semi-axis a_i directly in E_5.

An overview and more complete revision of these methods are proposed in the article by Balthazar and Malcher (2007).

3. A NEW METHOD BASED ON THE PRISMATIC HULL

In order to obtain a general measure of $\sqrt{J_{2,a}}$, we propose to generalize the method proposed by Li and De Freitas (2002) to cases involving complex three-dimensional loads. We have therefore decided to retain the amplitudes evaluated along the principal axes of the loading path in E_5 and retain the norm in the assessment of $\sqrt{J_{2,a}}$.

For these axes, we propose to make an analysis of the eigenvectors of the matrix of "mean squares" $\mathbf{V}_S$ of $\mathbf{S}(t)$:

$$\mathbf{V}_S = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \mathbf{S} \cdot \mathbf{S}^T \mathrm{d}t$$

Denoting $\mathbf{R} = \lfloor R_1, R_2, R_3, R_4, R_5 \rfloor^T$, the amplitudes reached by the stresses along these axes, we obtain an unique measure of $\sqrt{J_{2,a}}$:

$$\sqrt{J_{2,a}} = \sqrt{R_1^2 + R_2^2 + R_3^2 + R_4^2 + R_5^2} = \|\mathbf{R}\| \tag{6}$$

The complete methodology can be then described as follows:

1. Transformation in E_5 of the stress history $\boldsymbol{\sigma}(t)$: $\mathbf{S}(t) = \mathbf{P}.\boldsymbol{\sigma}(t)$;
2. Evaluation of the matrix $\mathbf{V}_S$ (which can be easily evaluated for complex signals using the Fourier Transform);
3. Evaluation of $\mathbf{S}'(t) = \mathbf{T} \cdot \mathbf{S}(t) = [S_1'(t), S_2'(t), S_3'(t), S_4'(t), S_5'(t)]^T$, the vector associated to $\mathbf{S}(t)$ from an orthogonal transformation, such that $\mathbf{T}^T\mathbf{T} = \mathbf{I}$ ($\mathbf{I}$ being the identity matrix);
4. Assessment of $\mathbf{R}$ from the maxima and minima of $\mathbf{S}'(t)$: $\mathbf{R} = \frac{1}{2}\left[\max_{t\in T}(\mathbf{S}'(t)) - \min_{t\in T}(\mathbf{S}'(t))\right]$;
5. Calculation of the norm of the vector $\mathbf{R}$ to produce the measure of $\sqrt{J_{2,a}}$ from the eq. (6)

In the transformation of the third step, $\mathbf{T}$ is a matrix containing the five eigenvectors of $\mathbf{V}_S$. These five vectors are in fact the principal axes of the load. In addition, $\mathbf{T}$ can also be interpreted as a rotation matrix if for a suitable choice of signs of these vectors, $\det(\mathbf{T}) = 1$. We further note that the proposed methodology is direct and does not need to perform iterations.

To illustrate simply the method, let consider two current two-dimensional examples of the literature: a test of tensile shear (7) and a biaxial test (8) such as:

$$\sigma_{xx}(t) = \sigma_{xx,m} + \sigma_{xx,a}\sin(\omega t), \sigma_{xy}(t) = \sigma_{xy,m} + \sigma_{xy,a}\sin(\omega t - \delta_{xy}) \tag{7}$$

$$\sigma_{xx}(t) = \sigma_{xx,m} + \sigma_{xx,a}\sin(\omega t), \sigma_{yy}(t) = \sigma_{yy,m} + \sigma_{yy,a}\sin(\omega t - \delta_{yy}) \tag{8}$$

Under these loads, $\delta_\bullet$ are the phase shift between the two signals, while the quantities ($\sigma_{\bullet,a}, \tau_{\bullet,a}$) and ($\sigma_{\bullet,m}, \tau_{\bullet,m}$) correspond to the amplitudes and mean over time of $\underline{\underline{\sigma}}(t)$.

For the loading (7), we can write:

$$\mathbf{V}_S = \frac{1}{2}\begin{bmatrix} \frac{1}{3}\sigma_{xx,a}^2 & \frac{1}{\sqrt{3}}\sigma_{xx,a}\sigma_{xy,a}\cos\delta_{xy} \\ \frac{1}{\sqrt{3}}\sigma_{xx,a}\sigma_{xy,a}\cos\delta_{xy} & \sigma_{xy,a}^2 \end{bmatrix} \tag{9}$$

$$\mathbf{R} = \frac{1}{6}\begin{bmatrix} \sigma_{xx,a}^2 + \sigma_{xy,a}^2 - \sqrt{12\sigma_{xx,a}^2\sigma_{xy,a}^2\cos^2\delta_{xy} + \left(\sigma_{xx,a}^2 - 3\sigma_{xy,a}^2\right)^2} \\ \sigma_{xx,a}^2 + \sigma_{xy,a}^2 + \sqrt{12\sigma_{xx,a}^2\sigma_{xy,a}^2\cos^2\delta_{xy} + \left(\sigma_{xx,a}^2 - 3\sigma_{xy,a}^2\right)^2} \end{bmatrix}$$

and finally obtain $\sqrt{J_{2,a}} = \sqrt{\frac{1}{3}\sigma_{xx,a}^2 + \sigma_{xy,a}^2}$. Then we notice that the phase is not involved in this result. This is a finding that can be confirmed experimentally for mild steel or for a material such as 34Cr4 (Liu, 1991).

Following a development similar for the load (8), we have:

$$\mathbf{V}_S = \frac{1}{8}\begin{bmatrix} \frac{1}{3}\left(4\sigma_{xx,a}^2 - 4\sigma_{xx,a}\sigma_{yy,a}\cos\delta_{yy} + \sigma_{yy,a}^2\right) & \frac{1}{\sqrt{3}}\left(2\sigma_{xx,a}\sigma_{yy,a}\cos\delta_{yy} - \sigma_{yy,a}^2\right) \\ \frac{1}{\sqrt{3}}\left(2\sigma_{xx,a}\sigma_{yy,a}\cos\delta_{yy} - \sigma_{yy,a}^2\right) & \sigma_{yy,a}^2 \end{bmatrix}$$

$$\mathbf{R} = \frac{1}{6}\begin{bmatrix} \sigma_{xx,a}^2 + \sigma_{yy,a}^2 - \sigma_{xx,a}\sigma_{yy,a}\cos\delta_{yy} + \sqrt{\left(4\cos^2\delta_{yy} - 1\right)\sigma_{xx,a}^2\sigma_{yy,a}^2 + \sigma_{xx,a}^4 + \sigma_{yy,a}^4 - 2\cos\delta_{yy}\left(\sigma_{xx,a}^3\sigma_{yy,a} + \sigma_{xx,a}\sigma_{yy,a}^3\right)} \\ \sigma_{xx,a}^2 + \sigma_{yy,a}^2 - \sigma_{xx,a}\sigma_{yy,a}\cos\delta_{yy} - \sqrt{\left(4\cos^2\delta_{yy} - 1\right)\sigma_{xx,a}^2\sigma_{yy,a}^2 + \sigma_{xx,a}^4 + \sigma_{yy,a}^4 - 2\cos\delta_{yy}\left(\sigma_{xx,a}^3\sigma_{yy,a} + \sigma_{xx,a}\sigma_{yy,a}^3\right)} \end{bmatrix}$$

and finally $\sqrt{J_{2,a}} = \sqrt{\frac{1}{3}\left(\sigma_{xx,a}^2 + \sigma_{yy,a}^2\right) - \sigma_{xx,a}\sigma_{yy,a}\cos\delta_{yy}}$. In this case, the expression is function of the phase as showed by Papadopoulos in [12] with a maximum amplitude for $\delta_{yy} = 180°$.

We can verify that the proposed method allows us to obtain the same results as those in plane stress presented by Li and De Freitas (2002) for sinusoidal, triangular or square loads type. But it also produces the general result proposed by Papadopoulos (1995) for 3D problems:

$$\sqrt{J_{2,a}} = \sqrt{\tfrac{1}{3}\left(\sigma_{xx,a}^2 + \sigma_{yy,a}^2 + \sigma_{zz,a}^2 - \sigma_{xx,a}\sigma_{yy,a}\cos(\delta_{yy}) - \sigma_{xx,a}\sigma_{zz,a}\cos(\delta_{zz}) - \sigma_{yy,a}\sigma_{zz,a}\cos(\delta_{yy} - \delta_{zz})\right) + \sigma_{xy,a}^2 + \sigma_{xz,a}^2 + \sigma_{yz,a}^2} \tag{10}$$

Geometrically, this method generalizes therefore the elliptical hull of Li and De Freitas (2002) by an ellipsoid in E_5. However, this interpretation is correct only for sinusoidal and synchronous loads. Indeed, looking after the amplitudes of the principal axes of complex load paths (for example, an asynchronous loading as represented in the Fig. 1), it is more general to consider that this path is circumscribed by a prismatic hull. We can call this method "the method of Prismatic Hull in the Principal Repair".

4. ILLUSTRATIVE EXAMPLES

Let consider as a first illustrative example a load such as:

$$\sigma_{xx}(t) = \sigma_{xx,a}\sin(\omega t),$$
$$\sigma_{xy}(t) = \sigma_{xy,a}\sin(\lambda_{xy}\omega t - \delta_{xy})$$

with $\sigma_{xx,a} = 263\,\text{MPa}$, $\sigma_{xy,a} = 132\,\text{MPa}$ and $\lambda_{xy} = 4$. The projections in E_5 of the vector $\mathbf{S}'(t)$ are represented in the Fig. 1.(a), (b) and (c) for $\delta_{xy} = 0°$, $\delta_{xy} = 45°$ and $\delta_{xy} = 90°$ respectively. In this case, the matrix $\mathbf{V}_S$ is diagonal and similar to the one of (9) for this part. The values given on these figures are the measure of the semi-axes and the associated norms. We have then:

$$\mathbf{R} = \begin{bmatrix} R_1 = 132 \\ R_2 = 151.84 \end{bmatrix}$$

$$\sqrt{J_{2,a}} = \|\mathbf{R}\| = \sqrt{\tfrac{1}{3}\sigma_{xx,a}^2 + \sigma_{xy,a}^2} = 201.2\text{MPa} .$$

We can therefore conclude with this numerical result and its geometrical interpretation on the various figures that $\sqrt{J_{2,a}}$ is independent of the phase in this load case. This behavior may be typical of a ductile material such as GGG60 for example (Liu, 1991).

Let consider now as second example, a more complex loading with the following form:

$$\sigma_{xx}(t) = \sigma_{xx,a}\sin(\omega t),$$
$$\sigma_{xy}(t) = \sigma_{xy,a}\left[\sin(\omega t) + \sin(4\omega t)\right]$$

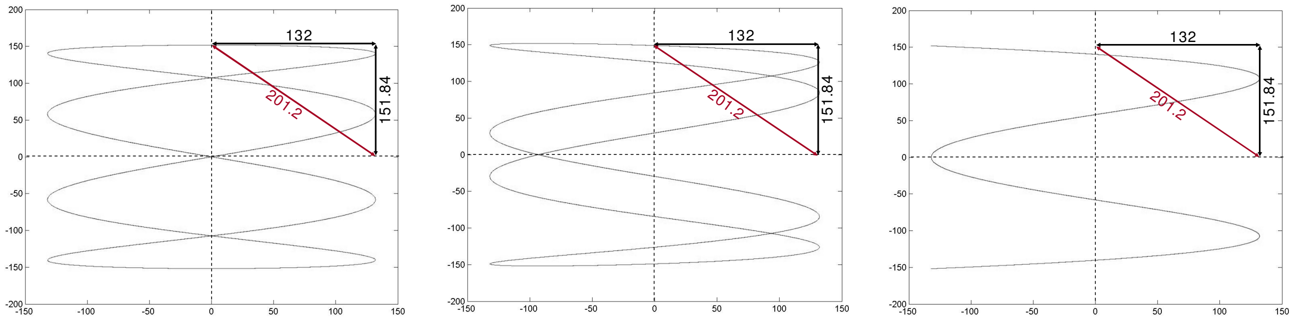
with $\sigma_{xx,a} = 200\,\text{MPa}$ and $\sigma_{xy,a} = 100\,\text{MPa}$. The projections in E_5 of the vectors $\mathbf{S}(t)$ and $\mathbf{S}'(t)$ are described in the Fig. 2.(a) and 2.(b). The values given on these figures are once again the measure of the semi-axes and the associated norms, and for this example, we have:

$$\mathbf{V}_S = \frac{1}{2}\begin{bmatrix} \frac{1}{3}\sigma_{xx,a}^2 & \frac{1}{\sqrt{3}}\sigma_{xx,a}\sigma_{xy,a} \\ \frac{1}{\sqrt{3}}\sigma_{xx,a}\sigma_{xy,a} & 2\left(\sigma_{xy,a}^2\right) \end{bmatrix}$$

$$\mathbf{R} = \begin{bmatrix} R_1 = 89.9 \\ R_2 = 219.03 \end{bmatrix}$$

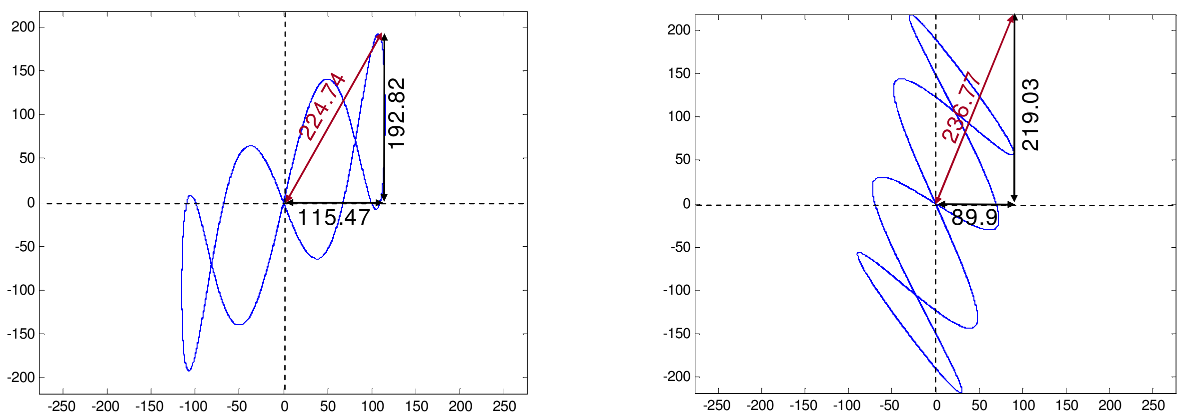
$$\sqrt{J_{2,a}} = \|\mathbf{R}\| = 236.77\ \text{MPa}$$

Thus, we see on this more complex situation of multiaxial and asynchronous loading, that the norm given by the maxima in the principal direction is greater than that given in E_5.



(a) (b) (c)

Fig. 1. Projections in E_5 of $\mathbf{S}'(t)$ for (a) $\delta_{xy} = 0°$ (b) $\delta_{xy} = 45°$ (c) $\delta_{xy} = 90°$



(a) (b)

Fig. 2. Projection in the five dimensions space of the vectors (a) $\mathbf{S}(t)$ and (b) $\mathbf{S}'(t)$

5. CONCLUSION

We have proposed in this work a new definition of the equivalent shear stress amplitude. This definition allows generalizing the method of Li and De Freitas (2002) in 3D. It allows to retrieve the results presented by Li and De Freitas for plane loads (sinusoidal or not) and also the general expression proposed by Papadopoulos (1995) for 3D problems. The operation of this method is illustrated by simple examples of tensile shear and biaxial tension.

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