CONSTRAINED UNSATURATED FLOWS THROUGH RIGID POROUS MEDIA

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Abstract. *This article introduces a new constitutive model for the partial pressure in unsaturated flows of incompressible liquids through rigid porous media and presents Riemann problem solutions accounting for the physical constraints associated to these phenomena. This constitutive equation for the pressure gives rise to a mathematical description in which the geometrical bound, representing the rigid porous medium assumption, is naturally taken into account – so that only physically meaningful solutions are allowed. A mixture theory approach describes the flow by considering three overlapping continuous constituents, representing the porous matrix (solid constituent), the fluid (liquid constituent) and an inert gas included to account for the compressibility of the mixture as a whole. Simulations employing the classical model and the new constitutive approach support the latter.*

Keywords*:* Constrained flow, unsaturated rigid porous media, shock waves, mixture theory.

1. INTRODUCTION

Unsaturated fluid flows through porous media are characterized by a strong dependence of the motion on the saturation, i.e. a force depending on the saturation gradient gives rise to the fluid flow. An adequate constitutive law, concerning rigid porous media, must account for the existence of an upper bound for the volume of the liquid that cannot exceed the volume of the pores – otherwise physically unrealistic solutions might be allowed.

The volume averaging technique widely used for modeling transport in porous media has been carefully reviewed by Alazmi and Vafai (2000). Other approaches, such as theories of consolidation (Biot, 1941; Lewis and Schrefler, 1998) and multi-scales (Hassanizadeh and Gray, 1980; 1990) models are largely used whenever porous media deformation is accounted for. A distinct approach – convenient for modeling multicomponent systems – is used in this work: a continuum theory of mixtures, supported by a local theory with thermodynamic consistency. Its basic assumption is that, at any time, all the constituents are present at every point of the mixture, which is composed by superimposed cinematically independent continuous constituents (Atkin and Craine, 1976; Rajagopal and Tao, 1995).

The mixture encompasses three constituents: a solid (modeling the porous matrix), a liquid (modeling an incompressible Newtonian liquid) and an inert gas (accounting for the compressibility of the system as a whole). The mechanical model is built in by assuming the porous medium homogeneous, rigid and at rest and the gas with zero mass density. In this case, the mechanical model is represented by and mass and momentum equations for the fluid constituent combined with convenient constitutive assumptions.

This work presents a constitutive relation for the (partial) pressure of the liquid constituent which accounts for the geometrical bound (arising from the rigidity of the porous matrix and the incompressibility of the liquid), besides assuring continuity for the pressure and for its first derivative, thus allowing analytical computation of the Riemann invariants associated to the problem – an important step for a simulation employing Glimm's scheme, specifically developed to treat discontinuous problems. Comparison of this proposed constitutive relation with the classical one – in which the partial pressure is a quadratic function of the fluid fraction – lend a strong support to the former, showing the shortcoming of the usual constitutive equation when describing phenomena involving unsaturated rigid porous media.

2. MECHANICAL MODEL

The mechanical model is built in by considering constitutive relations – namely the partial stress tensor and the momentum source – and mass and momentum equations for the liquid constituent only, since the solid matrix was assumed rigid and the gas was assumed inert. The fluid (liquid) constituent must satisfy (Atkin and Craine, 1976; Rajagopal and Tao, 1995) the following balance equations:

$$\begin{aligned} &\frac{\partial \rho_F}{\partial t} + \nabla \cdot (\rho_F \mathbf{v}_F) = 0 \\ &\rho_F \left[\frac{\partial \mathbf{v}_F}{\partial t} + (\nabla \mathbf{v}_F)\mathbf{v}_F \right] = \nabla \cdot \mathbf{T}_F + \mathbf{m}_F + \rho_F \mathbf{b}_F \end{aligned} \tag{1}$$

in which and $\mathbf{v}_F$ is the fluid constituent velocity in the mixture and ρ_F stands for the fluid constituent mass density – representing the local ratio between the fluid constituent mass and the corresponding volume of mixture. The partial stress tensor associated with the fluid constituent is given by $\mathbf{T}_F$, the body force per unit mass is represented by $\mathbf{b}_F$ (here $\mathbf{b}_F \equiv \mathbf{g}$) while $\mathbf{m}_F$ is the momentum supply acting on the fluid constituent due to its interaction with the remaining constituents of the mixture. This momentum source is an internal contribution; consequently the net momentum supply to the mixture – due to all the constituents – must be zero: $\sum_{i=1}^{n} \mathbf{m}_i = 0$.

The balance of angular momentum is satisfied through an adequate choice of $\mathbf{T}_F$, being automatically fulfilled whenever the partial stress tensor is assumed symmetrical. Since the flow is assumed isothermal, the energy balance is not considered.

Before presenting the constitutive relations, an important quantity must be considered. The fluid fraction φ is defined as the ratio between the fluid constituent mass density ρ_F and the actual mass density of the fluid ρ_f – regarded as a single continuum. In other words: $\varphi = \rho_F / \rho_f$.

The momentum source term – which accounts for the dynamic interaction among the constituents in a mixture representing an unsaturated flow of an incompressible Newtonian fluid through a homogeneous porous matrix – is represented by the following constitutive relation (Williams, 1978; Saldanha da Gama and Sampaio, 1987):

$$\mathbf{m}_F = -C\varphi^2 \mathbf{v}_F - D\varphi \nabla \varphi \tag{2}$$

in which C and D are positive constants. The first term at the right hand side represents the drag between the liquid and the porous matrix – the so-called darcian term, suggested by the classical Darcy law, while the second term represents the forces arising from the gradients of concentration.

The following physical constraint results from the liquid incompressibility and the rigid porous medium assumptions:

$$\varphi \le \varepsilon \tag{3}$$

where ε is the porosity (the local ratio between the active pores volume and the total volume). The ratio $\psi = \varphi / \varepsilon$ is called saturation and, from Eq. (3), $\psi \le 1$.

An analogy with the stress tensor acting on an incompressible Newtonian fluid within a Continuum Mechanics framework probably led Williams (1978) to consider the partial stress tensor acting on the fluid constituent as being proportional to the pressure acting on it and to the gradient of its velocity. A constitutive relation analogous to the usually employed for Cauchy stress tensor with such a behavior comes as a consequence. A further simplification has been later proposed by Allen (1986), who concluded that among the three distinct momentum transfer mechanisms in the mixture – namely: shear stresses, interphase tractions and momentum transfer through fluid drag on the porous matrix, the normal fluid stresses were dominant, the shear stresses and interphase tractions being negligible when compared to the fluid drag, leading to the following approximated relation for the partial stress tensor:

$$\mathbf{T}_F = -\bar{p}\mathbf{I} \qquad \bar{p} = \alpha\varphi^2 \tag{4}$$

where $\bar{p}$ is the pressure acting on the fluid (liquid) constituent, $\mathbf{I}$ is the identity tensor and α is a constant. The relation for $\bar{p}$ arises from the hypothesis that the actual pressure inside a pore depends, linearly, on the fluid fraction φ, provided that the saturation $\psi = \varphi / \varepsilon$ is small. Since this relation does not account for the rigidity of the porous matrix and the incompressibility of the fluid, it is unable to prevent physically inadmissible states during some simulations. Actually, Eq. (4) is adequate only when the inequality $\varphi \le \varepsilon$ is satisfied by means of a convenient choice of initial and boundary conditions (Martins-Costa and Saldanha da Gama, 2001; 2005).

Saldanha da Gama (2005) proposed a relationship accounting for the restriction $\varphi \le \varepsilon$ – namely an upper bound for the fluid fraction. The proposed equation is a constitutive relation between the pressure p and the fluid fraction φ, obtained by considering the inert gas as an ideal gas that experiences an isothermal process when $\varphi \to \varepsilon$, that the specific volume of the gas is proportional to its actual pressure and also using the incompressibility of the liquid and the definition of the fluid fraction φ. The pressure of the inert gas (regarded from a continuum mechanics approach) is given by: $p_g = \bar{c} / (\varepsilon - \varphi)$, with $\bar{c}$ being a positive constant. Since the pressure acting on the gas is the same pressure acting on the liquid when a continuum mechanics viewpoint is considered, the pressure acting on the liquid constituent, for $\varphi \to \varepsilon$ is given by $\bar{p} = \varphi\bar{c} / (\varepsilon - \varphi)$, thus allowing the following expression for the partial stress tensor:

$$\mathbf{T}_F = -\bar{p}\mathbf{I} \qquad \bar{p} = \alpha\varphi^2 + \beta\frac{\varphi}{\varepsilon-\varphi} \tag{5}$$

A new constitutive relation for the fluid constituent partial pressure is introduced in this work, which, besides avoiding the occurrence of solutions without physical meaning during simulations – by accounting for a geometrical bound, representing the rigid porous medium assumption (Saldanha da Gama, 2005) – assures continuity for the pressure and for its first derivative, thus allowing analytical computation of the Riemann invariants associated to the problem. This proposed constitutive relation is the strictly convex function $\bar{p} = \hat{\bar{p}}(\varphi)$ given by:

$$\bar{p} = \begin{cases} \alpha\varphi^2 & \text{for } 0 < \varphi \le \varphi_0 \\ \beta\dfrac{\varphi}{\varepsilon-\varphi} + \gamma & \text{for } \varphi_0 < \varphi \le \varepsilon \end{cases} \tag{6}$$

where α (α>0), β (β>0) and γ are constants and the fluid fraction φ is an always positive quantity smaller than the porosity ε. Also, the constant φ_0 is an always positive quantity.

The following conditions, relating the constants α , β, γ and φ_0 (or expressing β and γ as functions of α and φ_0), must be satisfied to ensure that the function $\bar{p}$ and its first derivative with respect to fluid fraction φ are continuous:

$$\begin{aligned} &\alpha\varphi_0^2 = \beta\frac{\varphi_0}{\varepsilon-\varphi_0} + \gamma \quad \Rightarrow \quad \gamma = \alpha\varphi_0^2 - \beta\frac{\varphi_0}{\varepsilon-\varphi_0} \\ &\frac{d}{d\varphi}\left(\alpha\varphi^2\right) = \frac{d}{d\varphi}\left(\frac{\beta\varphi}{\varepsilon-\varphi}\right) \quad \text{for} \quad \varphi = \varphi_0 \quad \Rightarrow \quad \beta = \frac{2\alpha\varphi_0\left(\varepsilon-\varphi_0\right)^2}{\varepsilon} \end{aligned} \tag{7}$$

The expression of γ assures a continuous function for the pressure on the fluid constituent while the expression of β assures continuity for the first derivative of $\bar{p}$. In this case, the constitutive relation stated in Eq. (6) may be rewritten as:

$$\bar{p} = \begin{cases} \alpha\varphi^2 & \text{for} \quad 0 < \varphi < \varphi_0 \\ \alpha\left\{\dfrac{2\varphi_0\left(\varepsilon-\varphi_0\right)^2}{\varepsilon}\left[\dfrac{\varphi}{\varepsilon-\phi} - \dfrac{\varphi_0}{\varepsilon-\varphi_0}\right] + \varphi_0^2\right\} & \text{for} \quad \varphi_0 \le \varphi < \varepsilon \end{cases} \tag{8}$$

Or, considering a single equation valid for the whole domain ($0 < \varphi < \varepsilon$), the proposed constitutive relation gives rise to the following expression for the partial stress tensor:

$$\mathbf{T}_F = -\bar{p}\mathbf{I}\,, \text{ with } \bar{p} = \alpha\left\{\frac{\varphi_0\left(\varepsilon-\varphi_0\right)^2}{\varepsilon}\left[\left(\frac{\varphi}{\varepsilon-\varphi} - \frac{\varphi_0}{\varepsilon-\varphi_0}\right) + \left|\frac{\varphi}{\varepsilon-\varphi} - \frac{\varphi_0}{\varepsilon-\varphi_0}\right|\right] + \frac{\varphi_0^2 + \varphi^2 - \left|\varphi_0^2 - \varphi^2\right|}{2}\right\} \tag{9}$$

The mechanical model may be stated by combining the balance equations, Eq. (1), and the constitutive assumptions, Eq. (2) and Eq. (9). Some simplifying assumptions lead to the one dimensional description of the problem: all the quantities depend only on the time t and on the position x, v is the only non-vanishing component of the fluid constituent velocity $\mathbf{v}_F$, in a horizontal flow, so that gravity effects may be neglected. Also, in order to better visualize the consequences of the porous matrix rigidity – included in the constitutive relation (9) – the darcian term and the diffusive one are neglected. The above-mentioned assumptions yield the following non-linear hyperbolic system of partial differential equations:

$$\begin{cases} \dfrac{\partial\varphi}{\partial t} + \dfrac{\partial}{\partial x}\left(\varphi v\right) = 0 \\ \dfrac{\partial}{\partial t}\left(\varphi v\right) + \dfrac{\partial}{\partial x}\left(\varphi v^2 + \bar{p}\right) = 0 \end{cases} \tag{10}$$

with $\bar{p}$ given by Eq. (9). It is important to notice that the partial pressure $\bar{p}$ is an increasing function of φ within the physically admissible range $0 < \varphi < \varepsilon$.

3. THE ASSOCIATED RIEMANN PROBLEM

The Riemann problem associated to Eq. (10) is the following initial data problem

$$\begin{gathered}\frac{\partial \varphi}{\partial t}+\frac{\partial}{\partial x}(\varphi v)=0\\ \frac{\partial}{\partial t}(\varphi v)+\frac{\partial}{\partial x}\left(\varphi v^2+\bar{p}\right)=0\\ \text{with}\quad (\varphi,v)=\begin{cases}(\varphi_L,v_L) & \text{for}\quad t=0,\ -\infty<x<0\\ (\varphi_R,v_R) & \text{for}\quad t=0,\quad 0<x<\infty\end{cases}\end{gathered}\tag{11}$$

where φ_L, φ_R, v_R and v_L are constants and $\bar{p}=\hat{\bar{p}}(\varphi)$ is a convex function given by Eq. (6) or Eq. (9).

The solution (in a generalized sense) of the Riemann problem (11) depends only on the ratio x/t, giving rise to

$$\begin{bmatrix}-\dfrac{x}{t} & 1\\ \bar{p}^{\backslash}-v^2 & -\dfrac{x}{t}+2v\end{bmatrix}\frac{d}{d\xi}\begin{bmatrix}\varphi\\ \varphi v\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}\quad\text{with}\quad\begin{cases}(\varphi,v)=(\varphi_L,v_L) & \text{for}\quad \xi\to-\infty\\ (\varphi,v)=(\varphi_R,v_R) & \text{for}\quad \xi\to\infty\end{cases}\tag{12}$$

where $\bar{p}^{\backslash}$ represents the always positive first derivative of $\bar{p}$ with respect to φ, given by

$$\bar{p}^{\backslash}=\begin{cases}2\alpha\varphi & \text{for}\quad 0<\varphi<\varphi_0\\ \beta\dfrac{\varepsilon}{(\varepsilon-\varphi)^2} & \text{for}\quad \varphi_0\le\varphi<\varepsilon\end{cases}\tag{13}$$

From the above equations, it comes that x/t corresponds exactly to the eigenvalues of the matrix $\begin{bmatrix}0 & 1\\ \bar{p}^{\backslash}-v^2 & 2v\end{bmatrix}$, provided the unknowns φ and v are smooth. On the other hand, the smoothness of φ and v, for $\xi\in(\xi_1,\xi_2)$, is ensured only when an eigenvalue can be set equal to x/t over this interval.

The two real eigenvalues of system (11) are given, in increasing order, by $\lambda_1=\hat{\lambda}_1(\varphi,v)=v-\sqrt{\bar{p}^{\backslash}}$ and $\lambda_2=\hat{\lambda}_2(\varphi,v)=v+\sqrt{\bar{p}^{\backslash}}$. In the regions of the plane x-t where φ and v are not constant, the solution (12) holds if the following equation is satisfied

$$\begin{gathered}-\lambda_i d\varphi+d(\varphi v)=0\quad\Rightarrow\quad -\left(v\mp\sqrt{\bar{p}^{\backslash}}\right)d\varphi+d(\varphi v)=0\\ \text{or}\quad\begin{cases}-\lambda_1 d\varphi+d(\varphi v)=0 & \text{where}\ \lambda_1=v-\sqrt{\bar{p}'}=x/t\\ -\lambda_2 d\varphi+d(\varphi v)=0 & \text{where}\ \lambda_2=v+\sqrt{\bar{p}'}=x/t\end{cases}\end{gathered}\tag{14}$$

The Riemann invariants R_1 and R_2, associated to the eigenvalues λ_1 and λ_2, are obtained from Eqs. (14), being given by: $R_1=\text{constant}=\int\left(\sqrt{\bar{p}^{\backslash}}/\varphi\right)d\varphi+v$ and $R_2=\text{constant}=-\int\left(\sqrt{\bar{p}^{\backslash}}/\varphi\right)d\varphi+v$.

So, the Riemann invariants may be expressed as $R_1=\lambda_1+\Theta(\varphi)$ and $R_2=\lambda_2-\Theta(\varphi)$, where the function $\Theta(\varphi)$ is given by

$$\Theta(\varphi)=\frac{d}{d\varphi}\left[\varphi\int\frac{\sqrt{\bar{p}^{\backslash}}}{\varphi}d\varphi\right]=\begin{cases}\sqrt{8\alpha}\varphi^{3/2}+C & \text{for}\quad 0<\varphi<\varphi_0\\ \sqrt{\dfrac{\beta}{\varepsilon}}\ln\left(\dfrac{\varphi}{\varepsilon-\varphi}\right)+\dfrac{\sqrt{\varepsilon\beta}}{\varepsilon-\varphi}+C & \text{for}\quad \varphi_0\le\varphi<\varepsilon\end{cases}\tag{15}$$

A necessary and sufficient condition for a 1-rarefaction connection between the left state and an intermediate state is that the first eigenvalue between these two states must be an increasing function of x/t. On the other hand, these two states will be connected by a 1-shock if, and only if, the jump conditions and the entropy conditions for these two states are satisfied (Smoller, 1983).

Analogously, the right state will be connected to the intermediate state by a 2-rarefaction if, and only if, between these two states, the second eigenvalue is an increasing function of *x*/*t*. Also, these two states will be connected by a 2-shock if, and only if, the jump conditions and the entropy conditions for these two states are satisfied.

Two states will be connected by a shock if the Rankine-Hugoniot jump conditions (Smoller, 1983) given by:

$$\frac{[\varphi v]}{[\varphi]} = \frac{\left[\varphi v^2 + \bar{p}\right]}{[\varphi v]} = s \tag{16}$$

where "[ ]" denotes the jump and s denotes the shock (discontinuity) speed, are satisfied as well as the entropy conditions (Smoller, 1983).

The states (φ_L, v_L) and (φ_*, v_*) will be connected by a 1-shock if they satisfy the jump conditions and the relations $s_1 < \hat{\lambda}_1(\varphi_L, v_L)$ and $\hat{\lambda}_1(\varphi_*, v_*) < s_1 < \hat{\lambda}_2(\varphi_*, v_*)$, while the states (φ_*, v_*) and (φ_R, v_R) will be connected by a 2-shock if they satisfy the jump conditions and the inequalities $s_2 > \hat{\lambda}_2(\varphi_R, v_R)$ and $\hat{\lambda}_2(\varphi_*, v_*) > s_2 > \hat{\lambda}_1(\varphi_*, v_*)$.

The convexity of $\bar{p}$ ensures the entropy conditions, provided the Rankine-Hugoniot jump conditions hold and $\varphi_L < \varphi_*$ (for the 1-shock) or $\varphi_* > \varphi_R$ (for the 2-shock).

Table 1 summarizes the above results considering four possible solutions.

Table 1. Riemann problem possible solutions.

φ conditions	Left and right states connected by	v conditions
$\varphi_L > \varphi_* < \varphi_R$	1-rarefaction/2-rarefaction	$v_L < v_* < v_R$
$\varphi_L < \varphi_* > \varphi_R$	1-shock/2-shock	$v_L > v_* > v_R$
$\varphi_L > \varphi_* > \varphi_R$	1-rarefaction/2-shock	$v_L < v_* > v_R$
$\varphi_L < \varphi_* < \varphi_R$	1-shock/2-rarefaction	$v_L > v_* < v_R$

At this point it is convenient to introduce the function $\Lambda(\varphi)$ defined as

$$\Lambda(\varphi) = \int_{\varphi_0}^{\varphi} \frac{\sqrt{\bar{p}^{\backslash}(\omega)}}{\omega} d\omega = \begin{cases} \sqrt{8\alpha\varphi} - \sqrt{8\alpha\varphi_0} & \text{for } \varphi \le \varphi_0 \\ \sqrt{\dfrac{\beta}{\varepsilon}} \ln\left(\dfrac{\phi(\varepsilon - \varphi_0)}{\phi_0(\varepsilon - \varphi)}\right) & \text{for } \varphi \ge \varphi_0 \end{cases} \tag{17}$$

with $\int_a^b \frac{\sqrt{\bar{p}^{\backslash}(\omega)}}{\omega} d\omega = \Lambda(b) - \Lambda(a)$

The possible solutions for the Riemann problem stated by Eq. (11) are conveniently summarized in Table 2.

Table 2. A priori solutions for the Riemann problem.

CONDITIONS	SOLUTION
$-\left\|\Lambda(\varphi_R)-\Lambda(\varphi_L)\right\| > v_L - v_R$	1-rarefaction/2-rarefaction
$\sqrt{(\bar{p}_R-\bar{p}_L)\left(\frac{1}{\varphi_L}-\frac{1}{\varphi_R}\right)} < v_L - v_R$	1-shock/2-shock
$(\varphi_L-\varphi_R)\sqrt{\left(\frac{\bar{p}_R-\bar{p}_L}{\varphi_R-\varphi_L}\right)\frac{1}{\varphi_R\varphi_L}} > v_L - v_R > \Lambda(\varphi_R)-\Lambda(\varphi_L)$	1-rarefaction/2-shock
$(\varphi_R-\varphi_L)\sqrt{\left(\frac{\bar{p}_R-\bar{p}_L}{\varphi_R-\varphi_L}\right)\frac{1}{\varphi_R\varphi_L}} > v_L - v_R > -\Lambda(\varphi_R)+\Lambda(\varphi_L)$	1-shock/2-rarefaction
$v_L - v_R = \Lambda(\varphi_R)-\Lambda(\varphi_L)$	1-rarefaction
$v_L - v_R = -\Lambda(\varphi_R)+\Lambda(\varphi_L)$	2-rarefaction
$(\varphi_R-\varphi_L)\sqrt{\left(\frac{\bar{p}_R-\bar{p}_L}{\varphi_R-\varphi_L}\right)\frac{1}{\varphi_R\varphi_L}} = v_L - v_R$	1-shock
$(\varphi_L-\varphi_R)\sqrt{\left(\frac{\bar{p}_R-\bar{p}_L}{\varphi_R-\varphi_L}\right)\frac{1}{\varphi_R\varphi_L}} = v_L - v_R$	2-shock

Once the intermediate state (φ_*, v_*) is known the solution (φ, v) is given by:

$$\text{a) } 1-\text{rarefaction}/2-\text{rarefaction} \;\Rightarrow\; (\varphi,v)=\begin{cases}(\varphi_L,v_L) & \text{if} \quad -\infty < x/t < \hat{\lambda}_1(\varphi_L,v_L)\\ (f_1,g_1) & \text{if} \quad \hat{\lambda}_1(\varphi_L,v_L) \le x/t \le \hat{\lambda}_1(\varphi_*,v_*)\\ (\varphi_*,v_*) & \text{if} \quad \hat{\lambda}_1(\varphi_*,v_*) < x/t < \hat{\lambda}_2(\varphi_*,v_*)\\ (f_2,g_2) & \text{if} \quad \hat{\lambda}_2(\varphi_*,v_*) \le x/t \le \hat{\lambda}_2(\varphi_R,v_R)\\ (\varphi_R,v_R) & \text{if} \quad \hat{\lambda}_2(\varphi_R,v_R) < x/t < \infty\end{cases} \tag{18}$$

$$\text{b) 1-shock/2-shock} \;\Rightarrow\; (\varphi,v)=\begin{cases}(\varphi_L,v_L) & \text{if} \quad -\infty < x/t < s_1\\ (\varphi_*,v_*) & \text{if} \quad s_1 < x/t < s_2\\ (\varphi_R,v_R) & \text{if} \quad s_2 < x/t < \infty\end{cases} \tag{19}$$

$$\text{c) } 1-\text{rarefaction}/2-\text{shock} \;\Rightarrow\; (\varphi,v)=\begin{cases}(\varphi_L,v_L) & \text{if} \quad -\infty < x/t < \hat{\lambda}_1(\varphi_L,v_L)\\ (f_1,g_1) & \text{if} \quad \hat{\lambda}_1(\varphi_L,v_L) \le x/t \le \hat{\lambda}_1(\varphi_*,v_*)\\ (\varphi_*,v_*) & \text{if} \quad \hat{\lambda}_1(\varphi_*,v_*) < x/t < s_2\\ (\varphi_R,v_R) & \text{if} \quad s_2 < x/t < \infty\end{cases} \tag{20}$$

$$\text{d) } 1-\text{shock}/2-\text{rarefaction} \;\Rightarrow\; (\varphi,v)=\begin{cases}(\varphi_L,v_L) & \text{if} \quad -\infty < x/t < s_1\\ (\varphi_*,v_*) & \text{if} \quad s_1 < x/t < \hat{\lambda}_2(\varphi_*,v_*)\\ (f_2,g_2) & \text{if} \quad \hat{\lambda}_2(\varphi_*,v_*) \le x/t \le \hat{\lambda}_2(\varphi_R,v_R)\\ (\varphi_R,v_R) & \text{if} \quad \hat{\lambda}_2(\varphi_R,v_R) < x/t < \infty\end{cases} \tag{21}$$

in which the functions f_1, g_1, f_2 and g_2, obtained from the Riemann invariants, depend on the ratio x/t being given by

$$\begin{aligned} f_1 &= \Theta^{-1}\left(\hat{\lambda}_1(\varphi_L,v_L)+\Theta(\varphi_L)-\frac{x}{t}\right) \quad ; \quad f_2 = \Theta^{-1}\left(-\hat{\lambda}_2(\varphi_R,v_R)+\Theta(\varphi_R)+\frac{x}{t}\right)\\ g_1 &= v_L+\Lambda(\varphi_L)-\Lambda(\varphi) \quad ; \quad g_2 = v_R-\Lambda(\varphi_R)+\Lambda(\varphi)\end{aligned} \tag{22}$$

within the considered intervals.

4 EXAMPLES

In this section the importance of the proposed constitutive model is illustrated by particular examples of Riemann problems with solution given shock-1/shock-2, being explicitly shown that, in this case, the constitutive relation proposed here avoids the occurrence of solutions without physical sense. This example is obtained by considering the following initial data problem:

$$\begin{aligned} &\frac{\partial\varphi}{\partial t}+\frac{\partial}{\partial x}(\varphi v)=0\\ &\frac{\partial}{\partial t}(\varphi v)+\frac{\partial}{\partial x}\left(\varphi v^2+\overline{p}\right)=0\\ &\text{with} \quad (\varphi,v)=\begin{cases}(\breve{\varphi},\breve{v}) & \text{for} \quad t=0, \; -\infty<x<0\\ (\breve{\varphi},-\breve{v}) & \text{for} \quad t=0, \quad 0<x<\infty\end{cases}\end{aligned} \tag{23}$$

where $\breve{\varphi}\,(\breve{\varphi}<\varepsilon)$ and $\breve{v}$ are positive constants and $\overline{p}=\hat{\overline{p}}(\varphi)$ is given by Eqs. (6)-(9).

Since $\varphi_L=\varphi_R=\breve{\varphi}$ and $\breve{v}=v_L>v_R=-\breve{v}$ as stated in Eq. (23) it may be noticed from table 2 that the solution is shock-1/shock-2. The intermediate state, in this case, is obtained from

$$v_* = v_L-(\varphi_*-\varphi_L)\sqrt{\left(\frac{\overline{p}_*-\overline{p}_L}{\varphi_*-\varphi_L}\right)\frac{1}{\varphi_*\varphi_L}} \quad \text{and} \quad v_* = v_R-(\varphi_R-\varphi_*)\sqrt{\left(\frac{\overline{p}_R-\overline{p}_*}{\varphi_R-\varphi_*}\right)\frac{1}{\varphi_R\varphi_*}} \tag{24}$$

with φ_* being the unique root of

$$\breve{v}=(\varphi_*-\breve{\varphi})\sqrt{\left(\frac{\overline{p}-\overline{p}_*}{\breve{\varphi}-\varphi_*}\right)\frac{1}{\breve{\varphi}\varphi_*}} \tag{25}$$

with $\overline{p}$ stated in Eq. (6) .

The 1-shock/2-shock solution presented in Eq. (34), in this case, is reduced to

$$(\varphi,v)=\begin{cases}(\breve{\varphi},\breve{v}) & \text{if} \quad -\infty<x/t<s_1\\ (\varphi_*,0) & \text{if} \quad s_1<x/t<s_2\\ (\breve{\varphi},-\breve{v}) & \text{if} \quad s_2<x/t<\infty\end{cases} \tag{26}$$

where the shock speeds are

$$s_1 = -\frac{\breve{\varphi}\,\breve{v}}{\varphi_* - \breve{\varphi}} \quad \text{and} \quad s_2 = \frac{\breve{\varphi}\,\breve{v}}{\varphi_* - \breve{\varphi}} \tag{27}$$

Figures 1 to 4 show examples of Riemann problem solution with connections shock-1 and shock-2, by plotting the fluid fraction φ versus the velocity x/t. In all considered cases the porosity is ε=0.75 and four distinct situations are compared – namely unconstrained (no prescribed value for φ_0, which is equivalent to consider the classical constitutive relation for the pressure, given by Eq. (4)), φ_0=0.675, φ_0=0.450 and φ_0=0.225. In all cases the initial data gives rise to a zero intermediate velocity (v_*=0).

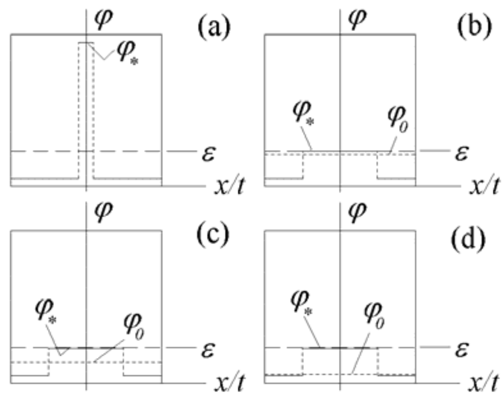

Figure 1. Riemann problem– φ vs x/t, with ε=0.75, α=0.10, $\breve{\varphi}=0.7$, $\breve{v}=2.0$: (a) unconstrained (no φ_0), φ_*=2.94, s_2=-s_1=0.146; (b) φ_0=0.675, φ_*=0.749, s_2=-s_1=0.728; (c) φ_0=0.450, φ_*=0.743, s_2=-s_1=0.737; (d) φ_0=0.225, φ_*=0.739, s_2=-s_1=0.742.

In Fig. 1, the constant α=0.10 and the initial fluid fraction and velocity are $\breve{\varphi}=0.7$ and $\breve{v}=2.0$. The unconstrained case (a) gives rise to shock speeds s_2=-s_1=0.146 and intermediate fluid fraction φ_*=2.94, violating inequality (3), since $\varphi_*>\varepsilon$. In cases (b), (c) and (d), a fluid fraction φ_0 has been imposed respectively as φ_0=0.675, φ_0=0.450 and φ_0=0.225 giving rise to φ_*=0.749, φ_*=0.743 and φ_*=0.739 and s_2=-s_1=0.728, s_2=-s_1=0.737 and s_2=-s_1=0.742, respectively, all cases satisfying inequality (3).

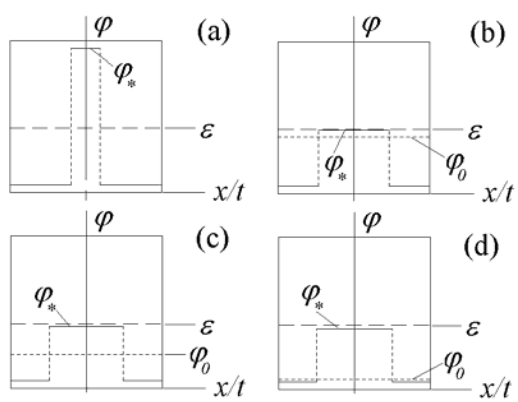

Figure 2. Riemann problem– φ vs x/t, with ε=0.75, α=0.10, $\breve{\varphi}=0.2$, $\breve{v}=1.0$: (a) unconstrained (no φ_0), φ_*=1.530, s_2=-s_1=0.150; (b) φ_0=0.675, φ_*=0.747, s_2=-s_1=0.366; (c) φ_0=0.450, φ_*=0.722, s_2=-s_1=0.383; (d) φ_0=0.225, φ_*=0.709, s_2=-s_1=0.393.

In Fig. 2 the constant α=0.10 (the same value employed in Fig. 1) and the initial fluid fraction and velocity are $\breve{\varphi}=0.2$, $\breve{v}=1.0$. The unconstrained case (a) gives rise to s_2=-s_1=0.150 and φ_*=1.530, also violating Eq. (3). In cases (b), (c) and (d), imposing respectively φ_0=0.675, φ_0=0.450 and φ_0=0.225, gives rise to φ_*=0.747, φ_*=0.722 and φ_*=0.709 and s_2=-s_1=0.366; s_2=-s_1=0.383 and s_2=-s_1=0.393, respectively, satisfying inequality (3).

In Fig. 3 the α -constant is ten times greater than the one employed in Figs. 1 and 2, α=1.0 and the initial fluid fraction and velocity are $\breve{\varphi} = 0.2$ and $\breve{v} = 2.0$. The unconstrained case (a) gives rise to s_2=-s_1=0.489 and φ_*=1.018, violating Eq. (3), like the two preceding figures. In cases (b), (c) and (d), for imposed fluid fractions φ_0=0.675, φ_0=0.450 and φ_0=0.225, it comes that φ_*=0.740, φ_*=0.685, φ_*=0.660 and, and s_2=-s_1=0.740, s_2=-s_1=0.8253 and s_2=-s_1=0.870, respectively, also satisfying inequality (3).

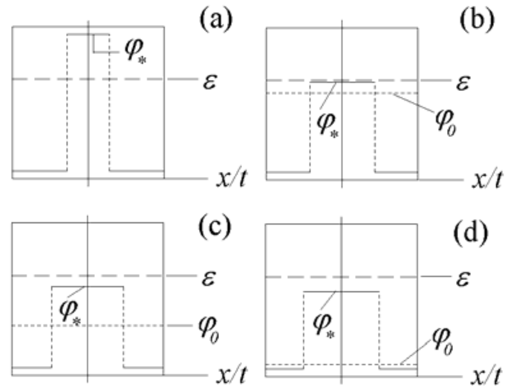

Figure 3. Riemann problem– φ vs x/t, with ε=0.75, α=1.0, $\breve{\varphi} = 0.2$, $\breve{v} = 2.0$: (a) unconstrained (no φ_0), φ_*=1.018, s_2=-s_1=0.489; (b) φ_0=0.675, φ_*=0.740, s_2=-s_1=0.740; (c) φ_0=0.450, φ_*=0.685, s_2=-s_1=0.8253; (d) φ_0=0.225, φ_*=0.6609, s_2=-s_1=0.870.

Considering cases (b), (c) and (d) of Figs. 1 to 3, it may be noticed a weak dependence of the shock speed values and the intermediate fluid fraction values on φ_0, which has been chosen in order to avoid that a fluid fraction greater than the porosity, a physically non realistic value, since the porous matrix is assumed rigid and the liquid incompressible. This weak dependence emphasizes the importance of the constitutive hypothesis for the pressure proposed in this work.

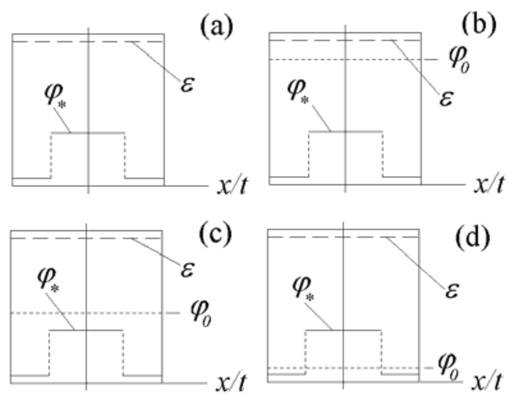

Figure 4. Riemann problem– φ vs x/t, with ε=0.75, α=1.0, $\breve{\varphi} = 0.2$, $\breve{v} = 0.5$: (a) unconstrained (no φ_0), φ_*=0.381, s_2=-s_1=0.552; (b) φ_0=0.675, φ_*=0.381, s_2=-s_1=0.552; (c) φ_0=0.450, φ_*=0.381, s_2=-s_1=0.552; (d) φ_0=0.225, φ_*=0.377, s_2=-s_1=0.566.

In Fig. 4 the α-constant is α=1.0 and the initial fluid fraction and velocity are $\breve{\varphi} = 0.2$, $\breve{v} = 0.5$. The unconstrained case (a) gives rise to s_2=-s_1=0.552 and φ_*=0.381 ($\varphi_* < \varepsilon$), satisfying the inequality (3). In cases (b), (c) and (d), considering φ_0=0.675, φ_0=0.450 and φ_0=0.225, gives rise to φ_*=0.381, φ_*=0.381 and φ_*=0.377, and s_2=-s_1=0.552; s_2=-s_1=0.552 and s_2=-s_1=0.56, respectively. Figure 4 shows that the proposed constitutive relation is able to simulate the problems where the inequality $\varphi \le \varepsilon$ is satisfied by means of a convenient choice of initial and boundary conditions – namely when the classical model for the partial pressure is employed, without accounting for the rigidity of the porous matrix and the incompressibility of the fluid. Also, in all cases – (a) to (d) – the intermediate fluid fraction and the shock speed values are very close, showing a weak dependence on φ_0.

7. FINAL REMARKS

The constitutive relation for the partial pressure proposed in this work accounts for the upper bound resulting from the rigidity of the porous matrix and provides continuity for the pressure and for its first derivative thus allowing the analytical computation of the Riemann invariants associated to the problem. This model avoids physically unrealistic solution occurring whenever the fluid fraction is allowed to be greater than the porosity. Besides, this constitutive model is convenient for numerical simulations employing Glimm's scheme, specifically designed for treating discontinuous problems, which preserves the shock identity.

A solution rarefaction1/rarefaction2 might be obtained, by considering initial data such that $\varphi_L = \varphi_R = \breve{\varphi}$ and $-\breve{v} = v_L < v_R = \breve{v}$, instead of $\varphi_L = \varphi_R = \breve{\varphi}$ and $\breve{v} = v_L > v_R = -\breve{v}$ as stated in Eq. (23). In this case, both unconstrained and constrained cases would satisfy inequality (3), since always $\varphi_* < \breve{\varphi}$.

As it was shown, the choice of φ_0 becomes an easy task since the solution is weakly dependent on it.

8. ACKNOWLEDGEMENTS

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