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SUBSPACE-BASED ALGORITHM FOR MODAL PARAMETERS IDENTIFICATION USING OUTPUT-ONLY DATA

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Abstract*. This paper deals with modal parameters identification using output-only data. A linear, time-invariant, finite dimensional mechanical system is considered, which is described by a stochastic state-space model excited by unknown operating forces. In this approach, the stochastic state-space model considers the errors due to state-variable and measurements, as integrant parts of the modeling, through a zero-mean white noise process. Subspace model identification consists in the approximation of a structured subspace described in terms of an extended observability matrix defined from a rank reduction process of a block Toeplitz matrix constructed from output-only data. This rank reduction procedure is calculated by using singular value decomposition. System matrices are evaluated using the shift-invariant property of the extended observability matrix. The natural frequencies, damping factors and modal shapes are identified by means of eigenvalues and eigenvectors of the state matrix, respectively. The performance of the presented approach is shown through simulation examples.*

Keywords*: modal parameters identification, output-only data, subspace-based algorithm, singular value decomposition*

1. INTRODUCTION

Mathematical modeling is an analytical approach used to describe the dynamic behavior of a phenomenon based on physical laws. System identification is an approach, where experiments are performed on the system, and a parametric model is subsequently fitted to the measured data by assigning a set of suitable numerical values to its parameters (Söderström and Stoica, 1987). Both approaches are important in system analysis, design and control problems. In the control community jargon, the process of fitting a state-space model to a multivariable linear time-invariant dynamic system from experimental data is called state-space realization (Viberg, 1995). A state-space is minimal if there exists no other realization of a lower degree to represent the system. The problem addressed in this paper deals with the application of a minimum order state-space realization technique in modal parameter identification using output-only data. More specifically, a time domain multivariable subspace-based parametric technique is used to identify the modal parameters of a structural system by fitting a suitable observable part of a minimal state-space model for a finite number of output-only data.

Modal parameter identification techniques are, classically, based on the input-output relationships (Maia and Silva, 1997). In general terms, a modal identification experiment is performed by fixing the structure to a test bench and actuators are used to produce controlled types of input forces, which are required to match a theoretical linear dynamic model, covering a frequency range which is compatible with both the experimental setup and desired region of analytical interest. On the other hand, a very interesting problem can be formulated when the objective is to analyze the dynamical behavior of a structure under operating conditions, where reality differs from the ideal laboratory environment and the input forces are not known, or just impossible to be measured. Examples that can be included in this situation consist in automotive structures excited by engine forces, offshore structures subjected to the turbulent action of the swell, aircraft structures subjected to unmeasurable ambient excitation, or civil structures like a bridge subjected to wind and traffic conditions (Abdelghani et al., 1999).

According to Peeters and Roeck (1999), there are many methods used to identify systems excited by unknown inputs. Formally, for a completely unknown input, it can be assumed that the system is excited by a white Gaussian process. A linear time-invariant autoregressive with moving average (ARMAV) model is then fitted to the data, using a prediction method (Soderstrom and Stoica, 1987). The MA characteristic of such an approach, leads to a highly non-linear minimization problem in order to calculate the parameters of the model. The solution of such a problem has a very large computational cost, especially for the multivariable parameters case. If the MA terms are omitted, in order to reduce the computational coast, an ARV model can then be used and simple least squares optimization solution method can be applied. The problem with the least squares approach is the overparametrization of the model that is needed resulting in a number of spurious numerical modes that must be separated from the true modes of the system.

Alternatively, subspace-based system identification methods have been used to overcome the drawbacks of the traditional system identification techniques. In the present approach, a subspace method offers a reliable way to fit the extended observation part of a multivariable stochastic state-space model realization, by means of a rank reduction operation upon a block Toeplitz matrix formed from output-only data using singular values decomposition (SVD). Stochastic state-space model considers the errors due to state-variable and measurements as integrant part of modeling through a zero-mean white noise process. The computational effort in such a method is relatively small. No non-linear optimization scheme is used and the system order can be estimated in a simple way. The state matrix is evaluated using the shift-invariant property of an extended observability matrix. The natural frequencies, damping factors and modal shapes are identified by means of eigenvalues and eigenvectors of the state matrix.

In this paper, a subspace technique is implemented for the identification of mechanical systems using output-only data. The paper is organized as follows. Section 2 describes the basic model of mechanical systems, section 3 discusses stochastic state-space models, to be used in output-only systems identification, section 4 focuses on the state-space realization for output-only. The modal parameters estimation procedure is described in section 5. An example, based on numerical simulations, is presented in section 6. Section 7 draws the main conclusions to the paper.

2. MATHEMATICAL MODELING OF MECHANICAL SYSTEMS

This section presents the basics of mathematical models of finite-dimensional, linear and time-invariant (LTI) mechanical systems.

The equation of motion of a f degrees of freedom LTI mechanical system is represented by the following second order matrix differential equation,

$$\overline{\mathbf{M}}\,\ddot{\mathbf{z}}(t)+\overline{\mathbf{C}}\,\dot{\mathbf{z}}(t)+\overline{\mathbf{K}}\,\mathbf{z}(t)=\mathbf{f}(t) \tag{1}$$

where $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$, $\overline{\mathbf{K}}$ are, respectively, the mass, damping and stiffness matrices, all of dimension $f \times f$. Vectors $\mathbf{z}(t)$ and $\mathbf{f}(t)$, of dimension $f \times 1$ represent, respectively, the generalized displacement and external forces acting on the system.

Equation (1) can be expressed in an equivalent continuous time state-space form (Gountier et al., 1993) as,

$$\dot{\mathbf{x}}(t)=\overline{\mathbf{A}}\,\mathbf{x}(t)+\overline{\mathbf{B}}\;\mathbf{u}(t) \tag{2}$$

with matrices $\overline{\mathbf{A}}$ and $\overline{\mathbf{B}}$ of dimension $n \times n$, given by

$$\overline{\mathbf{A}}=\begin{bmatrix}\mathbf{0} & \mathbf{I}_f\\ -\overline{\mathbf{M}}^{-1}\overline{\mathbf{K}} & -\overline{\mathbf{M}}^{-1}\overline{\mathbf{C}}\end{bmatrix} \qquad \text{and} \qquad \overline{\mathbf{B}}=\begin{bmatrix}\mathbf{0}\\ \overline{\mathbf{M}}^{-1}\mathbf{U}_f\end{bmatrix}, \tag{3}$$

where $n = 2\,f$ is the the state-space system model order, $\mathbf{x}(t)=\{\mathbf{z}(t)\quad \dot{\mathbf{z}}(t)\}^T$ is the generalized state vector of dimension $n \times 1$. Vector $\mathbf{u}(t)$ of dimension $m \times 1$ represents the non-null elements of the input vector $\mathbf{f}(t)$. Matrix $\mathbf{U}_f$ of dimension $f \times m$ is the input selection matrix, such that $\mathbf{f}(t)=\mathbf{U}_f\,\mathbf{u}(t)$, $\mathbf{I}_f$ is the identity matrix of dimension $f \times f$, and $\mathbf{0}$ denotes null matrices of appropriate dimensions.

Equation (2) constitutes the continuous-time state-space model for a finite dimensional LTI mechanical system. Solution for the state vector $\mathbf{x}(t)$ at time t with an input $\mathbf{u}(t)$ and initial conditions $\mathbf{x}(t_0)$ is given by,

$$\mathbf{x}(t)=e^{\overline{\mathbf{A}}(t-t_0)}\mathbf{x}(t_0)+\int_{t_0}^{t} e^{\overline{\mathbf{A}}(t-\tau)}\overline{\mathbf{B}}\,\mathbf{u}(\tau)\,d\tau \tag{4}$$

Experimental input-output data in modal analysis is obtained at equally spaced discrete time intervals. The continuous-time state-space model therefore needs to be rewritten in terms of a discrete-time representation. Let Δt be a constant time sampling interval. Substitution of $t=(k+1)\Delta t$ and $t_0 = k\Delta t$ into Eq. (4) yields,

$$\mathbf{x}[(k+1)\Delta t]=e^{\overline{\mathbf{A}}\,\Delta t}\mathbf{x}(k\Delta t)+\int_{k\Delta t}^{(k+1)\Delta t} e^{\overline{\mathbf{A}}[(k+1)\Delta t-\tau]}\;\overline{\mathbf{B}}\,\mathbf{u}(\tau)\,d\tau \tag{5}$$

Assuming that term $\mathbf{u}(\tau)$, of Eq. (5), has the constant value $\mathbf{u}(\tau) = \mathbf{u}(k\Delta t)$ over the interval $k\Delta t \le \tau \le (k+1)\Delta t$, and performing a change of variable τ by $\bar{\tau} = (k+1)\Delta t - \tau$, leads to,

$$\mathbf{x}[(k+1)\Delta t] = e^{\overline{\mathbf{A}}\,\Delta t}\mathbf{x}(k\Delta t) + \left[\int_0^{\Delta t} e^{\overline{\mathbf{A}}\,\bar{\tau}}\, d\bar{\tau}\ \overline{\mathbf{B}}\right]\mathbf{u}(k\Delta t) \tag{6}$$

Definition of the discrete quantities $\mathbf{x}[(k+1)\Delta t] = \mathbf{x}(k+1)$, $\mathbf{u}(k) = \mathbf{u}(k\Delta t)$ and matrices,

$$\mathbf{A} = e^{\overline{\mathbf{A}}\Delta t}$$

and (7)

$$\mathbf{B} = \int_0^{\Delta t} e^{\overline{\mathbf{A}}\,\bar{\tau}} d\bar{\tau}\ \overline{\mathbf{B}}$$

allows for the description of a state-space form of the mechanical system in discrete-time through the following expression,

$$\mathbf{x}(k+1) = \mathbf{A}\ \mathbf{x}(k) + \mathbf{B}\ \mathbf{u}(k)\,. \tag{8}$$

The set of observation variables measured during the modal testing of a structure is written in the following form,

$$\mathbf{y}(k) = \mathbf{C}\ \mathbf{x}(k) \tag{9}$$

where the term $\mathbf{y}(k)$ is an output vector of dimension $l \times 1$, associated with l response measurements. Matrices **A**, **B** and **C**, with appropriate dimensions, are, respectively, the state matrix, the input and output influence matrices.

Matrix **A** of Eq. (8) can be expressed in terms of its n eigenvalues and eigenvectors,

$$\mathbf{A} = \mathbf{\Psi}\ \mathbf{\Lambda}\ \mathbf{\Psi}^{-1} \tag{10}$$

where matrix $\mathbf{\Lambda} = diag(z_j)$ of dimension $n \times n$, contains the eigenvalues z_j, $j = 1,\cdots,n$ of **A.** The columns of the modal matrix $\mathbf{\Psi}$ of dimension $n \times n$ are the corresponding eigenvectors.

The first line of Eq. (7) can be used, in order to calculate the modal parameters of a flexible structure, yielding also a relationship between the state matrices $\overline{\mathbf{A}}$ and **A** of the continuous and discrete formulations. The eigenvalues in the two representations are related as.

$$\lambda_j = \log(z_j)/\Delta t \tag{11}$$

The natural frequencies ω_j and damping factors ξ_j are calculated as (Maia and Silva, 1997),

$$\omega_j = |\lambda_j| \qquad \text{and} \qquad \xi_j = -\mathrm{Re}al(\lambda_j)/|\lambda_j|, \tag{12}$$

where symbol $|\ |$ denotes absolute value. The mode shape $\boldsymbol{\varphi}_j$, associated to the j-th eigenvalue z_j, which is the observable part of the eigenvector $\mathbf{\Psi}_j$, is then trivially calculated through Eq. (9) as,

$$\boldsymbol{\varphi}_j = \mathbf{C}\ \mathbf{\Psi}_j \tag{13}$$

3. STOCHASTIC STATE-SPACE MODEL FOR OUTPUT-ONLY SYSTEMS

This section deals with stochastic state-space model to be used in output-only parameter identification. Stochastic components are included in the dynamic model, yielding the following deterministic-stochastic state-space model,

$$\mathbf{x}(k+1) = \mathbf{A}\ \mathbf{x}(k) + \mathbf{B}\ \mathbf{u}(k) + \mathbf{w}(k) \tag{14}$$

$$\mathbf{y}(k) = \mathbf{C}\ \mathbf{x}(k) + \mathbf{v}(k) \tag{15}$$

where vector $\mathbf{w}(k)$ of dimension $n \times 1$ and vector $\mathbf{v}(k)$ of dimension $l \times 1$ are the system and observation noises, respectively. The noise terms are assumed to be stationary white Gaussian processes with zero-mean and covariance matrices given by,

$$E\left[\begin{pmatrix}\mathbf{w}(k)\\ \mathbf{v}(k)\end{pmatrix}\begin{pmatrix}\mathbf{w}^T(j) & \mathbf{v}^T(j)\end{pmatrix}\right] = \begin{bmatrix}\mathbf{Q} & \mathbf{S}\\ \mathbf{S}^T & \mathbf{R}\end{bmatrix}\delta_{kj}, \tag{16}$$

where $E[.]$ denotes the statistical expectation and δ_{kj} is the Dirac delta function.

When only output signals are considered, Eqs. (14) and (15) assume the following form,

$$\mathbf{x}(k+1) = \mathbf{A}\,\mathbf{x}(k) + \mathbf{w}(k) \tag{17}$$

$$\mathbf{y}(k) = \mathbf{C}\,\mathbf{x}(k) + \mathbf{v}(k) \tag{18}$$

where, the unknown input term is implicitly incorporated to the noise terms $\mathbf{w}(k)$ and $\mathbf{v}(k)$.

In order to simplify the developments in the identification algorithm, the state vector of the system, $\mathbf{x}(k)$ is assumed to be a stochastic process with covariance matrix $\mathbf{\Sigma}$ of dimension $n \times n$ (Peeters and Roek, 1999), defined as,

$$E\left[\mathbf{x}(k)\cdot \mathbf{x}^T(k)\right] = \mathbf{\Sigma} \tag{19}$$

and which is independent of $\mathbf{w}(k)$ and $\mathbf{v}(k)$, i.e.,

$$E\left[\mathbf{x}(k)\cdot \mathbf{w}^T(k)\right] = \mathbf{0}$$
$$E\left[\mathbf{x}(k)\cdot \mathbf{v}^T(k)\right] = \mathbf{0} \tag{20}$$

Post-multiplying Eq. (17) by $\mathbf{x}^T(k+1)$, taking the mean value and considering Eqs. (16) and (19), leads to the following,

$$\mathbf{\Sigma} = E\left\{\left[\mathbf{A}\mathbf{x}(k) + \mathbf{w}(k)\right]\cdot\left[\mathbf{x}^T(k)\mathbf{A}^T + \mathbf{w}^T(k)\right]\right\} \tag{21.a}$$

or ,

$$\mathbf{\Sigma} = \mathbf{A}\,\mathbf{\Sigma}\,\mathbf{A}^T + \mathbf{Q} \tag{21.b}$$

The auto-covariance matrix of the output process $\mathbf{y}(k)$ and the cross-covariance matrix of the output process are defined as,

$$\mathbf{\Lambda}_i \equiv E\left[\mathbf{y}(k+i)\cdot \mathbf{y}^T(k)\right] \tag{22}$$

and

$$\mathbf{G} \equiv E\left[\mathbf{x}(k+1)\cdot \mathbf{y}^T(k)\right], \tag{23}$$

where matrix $\mathbf{\Lambda}_i$, valid for an arbitrary lag i, and $\mathbf{G}$, have respective dimensions, $l \times l$ and $n \times l$. Developing Eqs.(22) and (23), and using the assumptions made by Eqs. (19) and (20), allows to show that matrix $\mathbf{\Lambda}_i$ assumes the following form, for a lag $i \geq 1$:

$$\mathbf{\Lambda}_i = \mathbf{C}\mathbf{A}^{i-1}\mathbf{G}\,. \tag{24}$$

Equation (24) provides a relation between the system matrices (A, C) and the output auto-covariance matrix $\mathbf{\Lambda}_i$. This relation is important in establishing the state-space realization algorithm for output-only systems which is discussed in the next section.

4. STATE-SPACE REALIZATION FOR OUTPUT-ONLY SYSTEMS

This section presents an algorithm for modal parameters identification from a subspace-based state-space realization using output-only data. The present method consists in the estimation of matrices **A** and **C** of equations (17) and (18).

It is important to define coordinates of reference to calculate the mode shapes of the system, when dealing with modal parameters identification using only output data. The output vector of dimension $l \times 1$ is defined as,

$$\mathbf{y}(k) = \begin{Bmatrix} \mathbf{y}_r(k) \\ \mathbf{y}_s(k) \end{Bmatrix} \tag{25}$$

where $\mathbf{y}_r(k)$ is the reference output vector of dimension $r \times 1$. Vector $\mathbf{y}_s(k)$ of dimension $(l-r)\times 1$represents the non referenced outputs. The relation between $\mathbf{y}_r(k)$ and $\mathbf{y}_s(k)$,given by $\mathbf{y}_r(k) = \mathbf{L}\,\mathbf{y}(k)$ with $\mathbf{L} = [\mathbf{I}_r \quad \mathbf{0}]$ of dimension $r \times l$.

Covariance matrices between the complete output and state vector processes, and the reference vector, are defined as

$$\mathbf{\Lambda}_i^r = E\left[\mathbf{y}(k+i).\mathbf{y}_r^T(k)\right] = \mathbf{\Lambda}_i\,\mathbf{L}^T \tag{26}$$

and

$$\mathbf{G}^r = E\left[\mathbf{x}(k+1).\mathbf{y}_r^T(k)\right] = \mathbf{G}\,\mathbf{L}^T\,. \tag{27}$$

Matrices $\mathbf{\Lambda}_i^r$ and $\mathbf{G}^r$ have the respective dimensions $l \times r$ and $n \times r$. Matrices $\mathbf{\Lambda}_i^r$ and $\mathbf{G}^r$ are related, analogously to Eq. (24), through the following relation,

$$\mathbf{\Lambda}_i^r = \mathbf{C}\mathbf{A}^{i-1}\mathbf{G}^r\,. \tag{28}$$

A block Toeplitz matrix of dimension $il \times ir$, formed by matrices $\mathbf{\Lambda}_i^r$, is defined as,

$$\mathbf{T}^r = \begin{bmatrix} \mathbf{\Lambda}_i^r & \mathbf{\Lambda}_{i-1}^r & \cdots & \mathbf{\Lambda}_1^r \\ \mathbf{\Lambda}_{i+1}^r & \mathbf{\Lambda}_i^r & \cdots & \mathbf{\Lambda}_2^r \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Lambda}_{2i-1}^r & \mathbf{\Lambda}_{2i-2}^r & \cdots & \mathbf{\Lambda}_i^r \end{bmatrix} = \mathbf{Y}_s\,\mathbf{Y}_r^T\,, \tag{29}$$

where the block Hankel matrices $\mathbf{Y}_r$ and $\mathbf{Y}_s$ of dimensions $ri \times j$ and $li \times j$, respectively, are defined as,

$$\begin{pmatrix} \mathbf{Y}_r \\ \hline \mathbf{Y}_s \end{pmatrix} = \left(\frac{1}{\sqrt{j}}\right) \left(\begin{array}{cccc} \mathbf{y}_r(0) & \mathbf{y}_r(1) & \cdots & \mathbf{y}_r(j-1) \\ \mathbf{y}_r(1) & \mathbf{y}_r(2) & \cdots & \mathbf{y}_r(j) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}_r(i-1) & \mathbf{y}_r(i) & \cdots & \mathbf{y}_r(i+j-2) \\ \hline \mathbf{y}(i) & \mathbf{y}(i+1) & \cdots & \mathbf{y}(i+j-1) \\ \mathbf{y}(i+1) & \mathbf{y}(i+2) & \cdots & \mathbf{y}(i+j) \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{y}(2i-1) & \mathbf{y}(2i) & \cdots & \mathbf{y}(2i+j-2) \end{array} \right) \tag{30}$$

Equality (29) can be easily verified substituting Eqs. (28) and (30) into Eq. (29). Moreover, it is also simple to verify, by substituting Eq. (28) in Eq. (29) that,

$$\mathbf{T}^r = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{i-1} \end{bmatrix} \left[\mathbf{A}^{i-1}\mathbf{G}^r \quad \mathbf{A}^{i-2}\mathbf{G}^r \quad \cdots \quad \mathbf{AG}^r \quad \mathbf{G}^r\right] = \mathbf{\Gamma}_i\,\mathbf{\Lambda}_i^r \tag{31}$$

where $\mathbf{\Lambda}_i^r$ is a matrix of dimension $n \times ri$ and $\mathbf{\Gamma}_i$ is the observability matrix of dimension $il \times n$, formed by the state matrix **A** and the output influence matrix **C** as,

$$\mathbf{\Gamma}_i = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \mathbf{CA}^2 \\ \vdots \\ \mathbf{CA}^{i-1} \end{bmatrix}. \tag{32}$$

An important shift-invariant feature can be extracted from the observability matrix $\mathbf{\Gamma}_i$ of Eq. (32), as,

$$\mathbf{\Gamma}_i^{(2)} = \mathbf{\Gamma}_i^{(1)} \mathbf{A}, \tag{33}$$

where sub-matrices $\mathbf{\Gamma}_i^{(1)}$ and $\mathbf{\Gamma}_i^{(2)}$ are defined as,

$$\mathbf{\Gamma}_i = \begin{bmatrix} \mathbf{\Gamma}_i^{(1)} \\ \mathbf{CA}^{i-1} \end{bmatrix} = \begin{bmatrix} \mathbf{C} \\ \mathbf{\Gamma}_i^{(2)} \end{bmatrix}. \tag{34}$$

Assuming the realization to be of minimal order *n*, it follows that the observability matrix $\mathbf{\Gamma}_i$ is of full rank *n*. The state matrix **A** can be then derived from Eq. (33) as,

$$\mathbf{A} = \mathbf{\Gamma}_i^{(1)+} \mathbf{\Gamma}_i^{(2)}. \tag{35}$$

where the symbol “+ “denotes the Moore-Penrose pseudo-inverse. Matrix **C** is obtained from the first block row matrix $\mathbf{\Gamma}_i$.

5. MODAL PARAMETERS ESTIMATION USING THE SUB-SPACE METHOD

The column space from the product of matrices $\mathbf{T}^r = \mathbf{\Gamma}_i \mathbf{\Lambda}_i^r$, given by Eq. (31), is contained in the column space of $\mathbf{\Gamma}_i$. The row space of $\mathbf{T}^r$ is also contained in the row space of $\mathbf{\Lambda}_i^r$.

The rank of both matrices $\mathbf{\Gamma}_i$ and $\mathbf{\Lambda}_i^r$ is *n*, which is also the system order for the ideal case of noise-free output data, and assuming that $(li, ri) \geq n$. This makes the product $\mathbf{T}^r = \mathbf{\Gamma}_i \mathbf{\Lambda}_i^r$ of Eq. (31) to be also order of *n*. Moreover, the *n* columns of matrix $\mathbf{\Gamma}_i$ and the *n* rows of $\mathbf{\Lambda}_i^r$ span, respectively, the column and row spaces of $\mathbf{T}^r$, so that the column space of $\mathbf{T}^r$ has the same shift-invariant structure as that of $\mathbf{\Gamma}_i$, as expressed in Eq. (33).

In a more realistic case, where data is contaminated by noise, $\mathbf{T}^r$ is full rank. However, a rank *n* column space of $\mathbf{T}^r$ can be calculated from the following SVD partition as,

$$\mathbf{T}^r = \begin{bmatrix} \hat{\mathbf{Q}}_s & \hat{\mathbf{Q}}_n \end{bmatrix} \begin{bmatrix} \hat{\mathbf{S}}_s & \mathbf{0} \\ \mathbf{0} & \hat{\mathbf{S}}_n \end{bmatrix} \begin{bmatrix} \hat{\mathbf{V}}_s^T \\ \hat{\mathbf{V}}_n^T \end{bmatrix} = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s \hat{\mathbf{V}}_s^T + \hat{\mathbf{Q}}_n \hat{\mathbf{S}}_n \hat{\mathbf{V}}_n^T, \tag{36}$$

where matrices $\hat{\mathbf{Q}}_s$, $\hat{\mathbf{S}}_s$ and $\hat{\mathbf{V}}_s$ have dimensions $il \times n$, $n \times n$ and $ir \times n$, respectively. The condition $\hat{\mathbf{S}}_n = 0$ can be also verified in the absence of noise.

The *n* columns of matrix $\hat{\mathbf{Q}}_s$ span the column space of the *n*-order rank-reduced matrix $\hat{\mathbf{T}}^r = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s \hat{\mathbf{V}}_s^T$, recovered from a truncated SVD of $\mathbf{T}^r$ as described in Eq. (36). Such columns contain also the *n* principal left singular vectors corresponding to the *n* principal singular values of the diagonal matrix $\hat{\mathbf{S}}_s$.

In practice, the order *n* of the dynamical system can be selected via inspection of the number of the most significant singular values of $\mathbf{T}^r$. An estimate of the extended observability matrix, denoted by $\hat{\mathbf{\Gamma}}_i$, is thus taken as,

$$\hat{\mathbf{\Gamma}}_i = \hat{\mathbf{Q}}_s \qquad \text{or} \qquad \hat{\mathbf{\Gamma}}_i = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s^{1/2}. \tag{37}$$

The theoretical observability matrix $\mathbf{\Gamma}_i$ and the extended observability matrix $\hat{\mathbf{\Gamma}}_i = \hat{\mathbf{Q}}_s$ (or $\hat{\mathbf{\Gamma}}_i = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s^{1/2}$) span the column space of the data matrix $\mathbf{T}^r$ and $\hat{\mathbf{T}}^r$, respectively, for the ideal free-noise and noise contaminated cases, in an *n*-order state-space realization.

The extended observability matrix can be determined also as $\hat{\mathbf{\Gamma}}_i = \hat{\mathbf{Q}}_s \hat{\mathbf{S}}_s^{1/2}$. Note that the product of the columns of matrix $\hat{\mathbf{Q}}_s$ by the diagonal elements of matrix $\hat{\mathbf{S}}_s^{1/2}$ do not change the system poles, identified through Eq. (34).

Implementation of the present sub-space algorithm for modal parameters identification, based on output-only data, can be summarized in the following steps:

i) Estimation of the system order n via inspection of the number of the most significant singular values of matrix $\mathbf{T}^r = \mathbf{Y}_s \mathbf{Y}_r^T$, according to Eq. (29), for an appropriated choice of the number of columns data $\left(\frac{\mathbf{Y}_r}{\mathbf{Y}_s}\right)$, cf. Eq. (30).

ii) Extraction of the extended observability matrix $\hat{\mathbf{\Gamma}}_i$ from Eq. (36), using one of the two forms of Eq. (37) and the state matrix **A** described in Eq. (35).

iii) Estimation of the eigenvalues z_j and eigenvectors $\mathbf{\Psi}_j$ of the state matrix **A**, for $j = 1, \cdots, n$ (see Eq. 10).

iv) The n system's poles λ_j are calculated by Eq. (11). The natural frequencies ω_j and damping factors ξ_j are identified by Eq. (12).

v) Matrix **C** is finally obtained from the first block row of matrix $\hat{\mathbf{\Gamma}}_i$, cf. Eq. (34), and the mode shapes $\boldsymbol{\varphi}_j$ are estimated by Eq. (13).

6. EXAMPLE OF APPLICATION

In order to show the capabilities of the porposed subspace technique of modal parameters estimation, an example using simulated data is shown. The output set of data is obtained through numerical simulation of a seven degrees of freedom mass-spring-damper oscillator, as shown in Figure (1), using the following parameters: $m_1 = \cdots = m_7 = 1$ Kg, $c_1 = \cdots = c_7 = 5$ Ns/m and $k_1 = \cdots = k_7 = 2000$ N/m. Matrices $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ assume the following form,

$$\overline{\mathbf{M}} = \begin{bmatrix} m_1 & 0 & 0 & \cdots & 0 \\ 0 & m_2 & 0 & \cdots & 0 \\ 0 & 0 & m_3 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & 0 & m_7 \end{bmatrix} \tag{38}$$

$$\overline{\mathbf{C}} = \begin{bmatrix} c_1 + c_2 & -c_2 & 0 & \cdots & 0 \\ -c_2 & c_2 + c_3 & -c_3 & \cdots & 0 \\ 0 & -c_3 & c_3 + c_4 & \cdots & \vdots \\ \vdots & \vdots & \vdots & \ddots & -c_7 \\ 0 & 0 & 0 & -c_7 & c_7 \end{bmatrix} \tag{39}$$

$$\overline{\mathbf{K}} = \begin{bmatrix} k_1 + k_2 & -k_2 & 0 & \cdots & 0 \\ -k_2 & k_2 + k_3 & -k_3 & \cdots & \vdots \\ 0 & -k_3 & k_3 + k_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & -k_7 \\ 0 & 0 & 0 & -k_7 & k_7 \end{bmatrix} \tag{40}$$

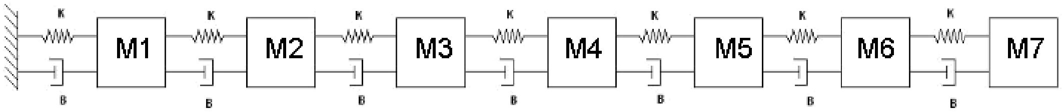

Figure 1. Seven degree of freedom oscillator system

Table 1 shows the exact natural frequencies and viscous damping factors for each mode of the system.

Table 1. Exact natural frequencies and damping factors.

Mode Number	Natural Frequency (Hz)	Damping Factor
1	2.1048	0.0052
2	4.9182	0.0155
3	7.9577	0.0250
4	10.6495	0.0335
5	12.8759	0.0405
6	14.5395	0.0457
7	15.5677	0.0489

The impulse response function (IRF) $h_{ij}(k) = h_{ij}(k\Delta t)$ is the response in output i at time $k\Delta t$, due to a unit impulse applied to input *j* at time 0. The mathematical expression for the IRF is easily derived from the parameters of the system as,

$$h_{ij}(k) = \sum_{l=1}^{n} \left[r_{ij(l)}\, e^{\lambda_l k\Delta t} + r_{ij(l)}^{*}\, e^{\lambda_l^{*} k\Delta t} \right] \tag{41}$$

where the term $r_{ij(l)} = \phi_{il}\,\phi_{jl}$ is the modal residue associated to eigenvalue λ_l (or z_l) and the terms ϕ_{ij}'s are elements of the mode shapes matrix $\mathbf{\Phi}$ wich is obtained from modal matrix $\mathbf{\Psi}$ as,

$$\mathbf{\Psi} = \begin{bmatrix} \mathbf{\Phi} & \mathbf{\Phi}^{*} \\ \mathbf{\Phi\Lambda} & \mathbf{\Phi}^{*}\mathbf{\Lambda}^{*} \end{bmatrix}. \tag{42}$$

Superscript * denotes complex conjugation and the eigenvalues λ_j's and modal matrix $\mathbf{\Phi}$ are calculated from matrices $\overline{\mathbf{M}}$, $\overline{\mathbf{C}}$ and $\overline{\mathbf{K}}$ as pointed out in Section 2.

A SIMO experiment is simulated, in order to obtain the output signals. A zero mean, white Gaussian noise signal with amplitude equal to 5 N is adopted as the input $u_1(k)$, actuating on mass 1 of the vibrating system shown in Fig. 1, the response $\mathbf{y}(k)$ is obtained through the following convolution operation,

$$y_i(k) = \sum_{j=0}^{N_p-1} h_{i1}(j)\, u_1(k-j) \qquad\qquad i = 1, \cdots, 7 \tag{43}$$

A parameters identification example is shown, in order to exemplify the performance of present method, considering only one reference ($r = 1$) and seven outputs ($l = 7$). The reference output in the present test corresponds to the output of mass 7 of the physical system shown in Fig. 1. A number of 400 data samples is adopted for the $\mathbf{y}(k)$ processes, available in a $i = 190$ rows and $j = 20$ columns data block, forming the Hankel matrices $\mathbf{Y}_r$ and $\mathbf{Y}_s$ of dimensions 190×20 and 1330×20, respectively, cf, Eq. (30), resulting in matrix $\mathbf{T}^r = \mathbf{Y}_s \mathbf{Y}_r^T$ of dimension 1330×190. The discretization interval Δt is 0.027 second.

It is important to remark that the rank of matrix $\mathbf{T}^r$ obtained in the identification process is 20 (equal the number *j* of columns of matrices $\mathbf{Y}_r$ and $\mathbf{Y}_s$), is different from the expected value of 7 (the system order *n*), according to the analysis made on Eq. (31). The reason for such a difference is attributed to the lack of an explicit input-output relationship present in the realization theory for parameters identification using output-only data and the simplified assumptions on the stochastic processes involving the state and output vectors made in sections 3 and 4.

The system order n is identified through inspection of the most significant singular values of matrix $\mathbf{T}^r$ as discussed in the section 5. Figure 1 shows a quantity of 20 important singular values of $\mathbf{T}^r$ from a total of 190 ones, agreeing with the rank of matrix $\mathbf{T}^r$ of the identification process. A careful inspection of Fig. 1, allows the observation of a gap between the first 6 singular values and the remaining ones. Such a fact suggests the adoption of the identified system order 6.

The system order adopted in this example is then 6. Appling the presented subspace method, it results in the identification of three first modes of the mechanical system. The identified natural frequencies and damping factors are shown in Tab. 2. Figures (2), (3) and (4) show the three identified mode shapes associated with the three first natural frequencies and damping factors as compared to exact modes derived from numerical simulation. The blue lines represents the exact modes and the red are the identified ones.

Taking into account the experience gained with running the identification code, it can be noted that higher order modes can also be identified by adopting a process based on repeatability of natural frequencies, damping factors and mode shapes. This can be done using successive values for system orders greater than 6 (as, for example, $n = 8, 10, 12, 14$ *and* 16).

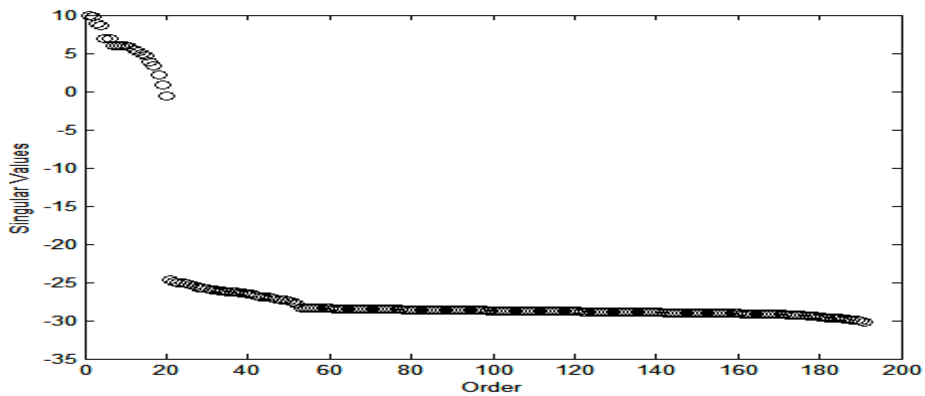

Figure 1. Singular values of matrix $\mathbf{T}^r$

Table 2. Exact and identified modal parameters.

Mode Number	Exact Natural Frequency (Hz)	Identified Natural Frequency (Hz)	Error (%)	Exact Damping Factor	Identified Damping Factor	Error (%)
1	1.6636	1.6631	0.0301	0.0052	0.0144	176.2308
2	4.9182	4.9501	0.6486	0.0155	0.0132	14.8387
3	7.9577	7.8315	1.5859	0.0250	0.0124	50.4000

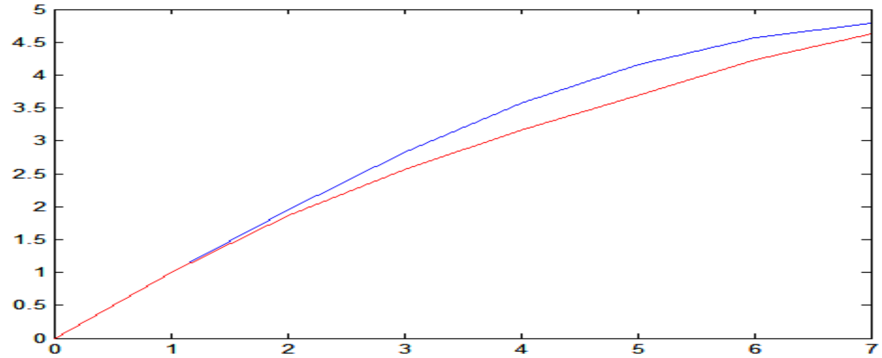

Figure 2. First mode shape (blue line: exact mode shape and red line: identified mode)

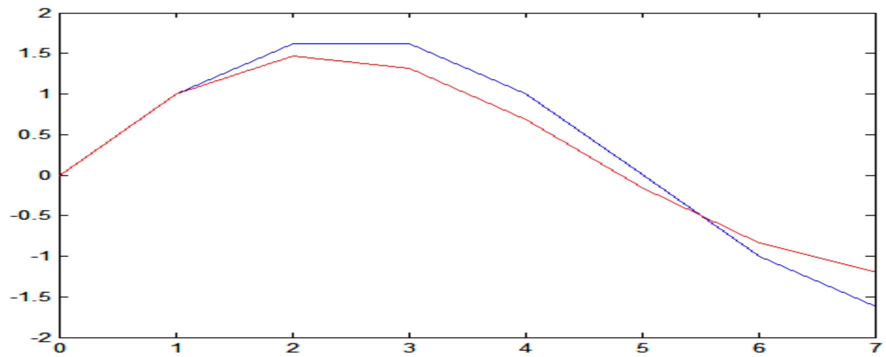

Figure 3. Second mode shape (blue line: exact mode shape and red line: identified mode)

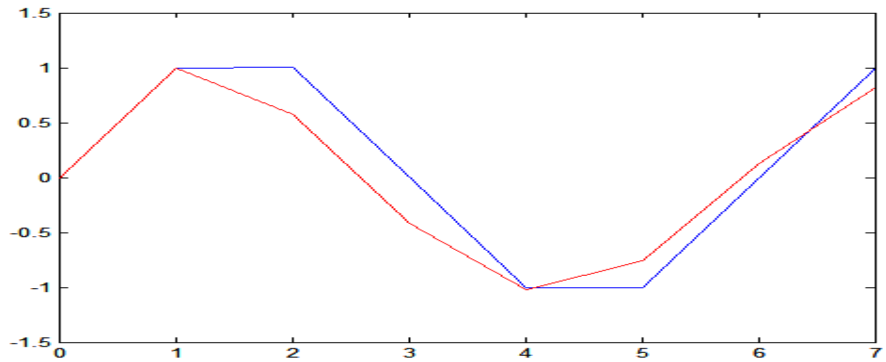

Figure 4. Third mode shape (blue line: exact mode shape and red line: identified mode)

7. CONCLUSION

The paper presents a subspace-based modal parameters identification algorithm valid for output-only data. The two main advantages of this approach are the compact formulation (only linear algebra is used) and relatively cheap computation (no optimization scheme is needed). Despite the fact that only output data is used, the results obtained through simulation are good. The success of the fit is attributed to the Gausian input data used in the computational simulation. These arguments encourage for the application of present method in practical situations as, for example, fault detection, modal analysis of civil structures or machines during operation condition, where the immeasurable ambient excitation agrees with a Gaussian process as suggested in the modeling of the present scheme. The references cited in this paper provides some examples of practical applications of present method.

8. REFERENCES

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9. RESPONSIBILITY NOTICE

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