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GEOMETRICAL DISCONTINUITIES IN STRESS UNILATERAL ELASTIC STRINGS

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Abstract. *Strings are ideally flexible one-dimensional continuous media exhibiting an unilateral behaviour: their internal efforts are always traction efforts. Under a compressive force, a string changes its geometry in such a way that the compression becomes traction. We consider the dynamical behaviour of stress-unilateral strings: the description of the motion of the string leads to a hyperbolic non linear system of conservation laws, which is linearly degenerated for small deformation analysis. We present an analysis of the propagation of the discontinuities in this situation: existence and propagation of geometrical discontinuities related to the stress-unilateral behaviour, tension and deformation waves. We present some numerical results furnished by a Godunov based method..*

Keywords*: unilateral problems, strings, non convex problems.*

1. Introduction

Strings are ideally flexible one-dimensional continuous media having the physical property that their internal efforts are tangent to their actual configuration. In addition, all the internal efforts correspond to tractions: under a compressive force, a string adapts its geometry in such a way that the compression becomes traction. Thus, strings are stress unilateral media.

The stress unilateral property of strings has been considered in the framework of static (Souza de Cursi, 1985; Souza de Cursi, 1987) or quasistatic problems (Souza de Cursi, 1990; Souza de Cursi, 1992). Extensions to bidimensional situations concerning fabrics (Schneider and Souza de Cursi, 1996; Souza de Cursi, 2004) and sail modelling (Le Maître *et al.*, 1997; Le Maître *et al.*, 1998a) can also be found in the literature. However, evolution problems taking into account the stress unilateral property remain incipient (Le Maître *et al.*, 1998b; Pego and Serre, 1988; Gilquin, 1989; Gilquin and Serre, 1989). This work is a step in this direction: the mechanical problem describing the evolution of a stress unilateral elastic string is written as a hyperbolic system and the propagation of the discontinuities is analyzed. The stress unilateral property introduces geometrical discontinuities connected to the unitary tangent to the configuration. The analysis of the propagation of the discontinuities is use to construct solutions of the mechanical problem.

2. Description of the motion of a stress unilateral string

In this section, we present the general equations describing the motion. We adopt the Lagrange's approach, where the quantities are brought to a fixed configuration.

2.1. Geometrical description in Lagrange's variables.

Strings are one-dimensional continuous media and the particles of the string are described by a single scalar variable. The natural choice for this scalar variable is the arc's length of the string, which may be taken on a natural configuration (Lagrange's approach) or on the actual configuration (Euler's approach). So, from the Lagrange's standpoint, the particles of the string are brought to a scalar $a \in (0, \ell) \subset R$, where ℓ is the natural length of the string (*i. e.*, its length at equilibrium with not any external forces applied) and R is the set of the real numbers. The position of each particle a is a point of the three dimensional space and is given by a vector $\mathbf{x} = (x_1, x_2, x_3)^t \in R^3$. The configurations of a string are curbs in the three dimensional space and the position of the particle a at the time t is $\mathbf{x}(a,t) = (x_1(a,t), x_2(a,t), x_3(a,t))^t \in R^3$. Thus, the position of the whole string is given by a function $\mathbf{x} : (0,\ell)\times(0,\tau) \to R^3$, where $(0, \tau)$ is the time interval under consideration.

Let us denote the derivatives by an index: for instance,

$$\mathbf{x}_a = \mathbf{x}' = \partial \mathbf{x}/\partial a \quad ; \quad \mathbf{x}_t = \dot{\mathbf{x}} = \partial \mathbf{x}/\partial t \quad ; \quad \mathbf{x}_{tt} = \ddot{\mathbf{x}} = \partial \dot{\mathbf{x}}/\partial t = \partial^2 \mathbf{x}/\partial t^2 \qquad (1)$$

The Euler's variable is denoted by s and corresponds to the arc length on the actual configuration $\mathbf{x}$. It is connected to the Lagrange's variable a by the relation

$$ds = (1+\varepsilon)da \quad ; \quad \varepsilon = |\mathbf{x}_a| - 1, \tag{2}$$

where the scalar ε is the deformation. The unitary tangent is the vector

$$\mathbf{t} = \mathbf{x}_a / |\mathbf{x}_a| \,. \tag{3}$$

2.2. Constitutive law

The internal efforts are given by a vector $\mathbf{T}=T\mathbf{t}$, where the scalar T is the tension. In the framework of infinitesimal deformations (but finite displacements are considered), the constitutive law reads as

$$T = k\varepsilon \qquad \text{and} \qquad T \geq 0\,, \tag{4}$$

where $k > 0$ is the elastic modulus. The equality corresponds to the standard Hooke's law and the inequality corresponds to the stress unilateral property. Equation (4) implies that $\varepsilon \geq 0$, *i. e.*, the length of the string can only increase, but not decrease. Thus, the length of every elementary part of the string is at least equal to the natural length and we have $ds \geq da$.

2.3. Conservation of the mass and equation of the motion in Lagrange's variables.

We denote by $\rho_0 > 0$ the natural linear mass density of the string and by $\rho = \rho(a,t) > 0$ the actual one. In the Lagrange's approach, the conservation of the mass reads as

$$\rho = (1+\varepsilon)\rho_0 \,. \tag{5}$$

The external distributed forces are given by $\rho\mathbf{g}(\mathbf{x},\mathbf{x}_t)$ and the motion of the string verifies

$$\frac{\partial \mathbf{T}}{\partial s} + \rho\mathbf{g}(\mathbf{x},\mathbf{x}_t) = \rho\mathbf{x}_{tt} \quad , \quad \left(0 < s < s(\ell)\,, 0 < t < T\right) \,. \tag{6}$$

By combining this equation with the conservation of the mass (Eq. (5)), we obtain the equation of the motion in Lagrange's variables:

$$\frac{\partial \mathbf{T}}{\partial a} + \rho_0\mathbf{g}(\mathbf{x},\mathbf{x}_t) = \rho_0\mathbf{x}_{tt} \quad , \quad \left(0 < a < \ell\,, 0 < t < T\right) \,. \tag{7}$$

2.4. Boundary and Initial Conditions.

The initial conditions are

$$\mathbf{x}(a,0) = \mathbf{x}_0(a) \quad ; \quad \mathbf{x}_t(a,0) = \dot{\mathbf{x}}_0(a) \quad (0 < a < \ell) \tag{8}$$

Typical boundary conditions are the mooring of the particle $a = 0$ and a force $\mathbf{f}(\mathbf{x}(\ell,t))$ applied to the particle $a = \ell$. These conditions reads as

$$\mathbf{x}(0,t) = 0 \quad ; \quad \mathbf{T}(\ell,t) = \mathbf{f}(\mathbf{x}(\ell,t)) \tag{9}$$

3. The hyperbolic system describing the motion of a string.

In this section, the equations of motion are reformulated as a quasilinear hyperbolic system and some properties concerning the propagation of the discontinuities are derived.

Let us introduce

$$\mathbf{u} = \mathbf{x}_a \quad , \quad \mathbf{v} = \mathbf{x}_t \tag{10}$$

$$\mathbf{U} = (\mathbf{x}, \quad \mathbf{u}, \quad \mathbf{v})^t = (x_1, x_2, x_3, u_1, u_2, u_3, v_1, v_2, v_3)^t \, . \tag{11}$$

Then,

$$\mathbf{g}(\mathbf{x}, \mathbf{x}_t) = \mathbf{g}(\mathbf{u}, \mathbf{v}) = \mathbf{G}(\mathbf{U}) \tag{12}$$

Moreover,

$$\mathbf{x}_t = \mathbf{u} \quad ; \quad \mathbf{u}_t = \mathbf{v}_a \quad ; \quad \mathbf{v}_t = \frac{k}{\rho_0}\left(\frac{|\mathbf{u}| - 1}{|\mathbf{u}|}\mathbf{u}\right)_a + \mathbf{G}(\mathbf{U}) \tag{13}$$

and Eq. (7) reads as follows:

$$\mathbf{U}_t + \mathbf{A}(\mathbf{U})\mathbf{U}_a + \mathbf{B}(\mathbf{U}) = 0 \quad , \quad (0 < a < \ell \, , 0 < t < T) \, , \tag{14}$$

where

$$\mathbf{A}(\mathbf{U}) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & \alpha_{11}(\mathbf{u}) & \alpha_{12}(\mathbf{u}) & \alpha_{13}(\mathbf{u}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{21}(\mathbf{u}) & \alpha_{22}(\mathbf{u}) & \alpha_{23}(\mathbf{u}) & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha_{31}(\mathbf{u}) & \alpha_{23}(\mathbf{u}) & \alpha_{33}(\mathbf{u}) & 0 & 0 & 0 \end{pmatrix} ; \; \mathbf{B}(\mathbf{U}) = \begin{pmatrix} -U_7 \\ -U_8 \\ -U_9 \\ 0 \\ 0 \\ 0 \\ -G_1(\mathbf{U}) \\ -G_2(\mathbf{U}) \\ -G_3(\mathbf{U}) \end{pmatrix} ; \tag{15}$$

$$\alpha_{ij}(\mathbf{U}) = -\frac{k}{\rho_0}\left(\frac{|\mathbf{u}| - 1}{|\mathbf{u}|}\right)\delta_{ij} - \frac{k}{\rho_0}\frac{u_i u_j}{|\mathbf{u}|^3} \quad . \tag{16}$$

Here, δ_{ij} denotes Kronecker's symbol (δ_{ij} =1 for i=j and δ_{ij} =0 otherwise). The eigenvalues and eigenvectors of $\mathbf{A}(\mathbf{U})$ are given in Table 1. Since

$$\nabla\lambda_i.\mathbf{E}_i = \sum_{j=1}^{9}\frac{\partial\lambda_i}{\partial U_j}E_{i,j} = 0 \quad (1 \le i \le 9), \tag{17}$$

we have the results 3.1 and 3.2 below (Jeffrey, 1976 ; Leroux and Schatzman, 1981).

Theorem 3.1: The system given in Eq. (14) is hyperbolic linearly degenerate for $|\mathbf{u}| \ge 1$ and ultrahyperbolic for $|\mathbf{u}| < 1$.

Theorem 3.2: All the admissible discontinuities are *contact discontinuities*. Moreover, the velocity of propagation of any discontinuity coincides with one of the eigenvalues of $\mathbf{A}(\mathbf{U})$.

Table 1. Eigenvalues of $\mathbf{A}(\mathbf{U})$.

Eigenvalue	Multiplicity	Example of associated eingenvector
$\lambda_1 = -\sqrt{\frac{k}{\rho_0}}$	1	$\mathbf{E}_1 = (0 \quad 0 \quad 0 \quad u_1 \quad u_2 \quad u_3 \quad -\lambda_1 u_1 \quad -\lambda_1 u_2 \quad -\lambda_1 u_3)$
$\lambda_2 = \lambda_3 = -\sqrt{\frac{k}{\rho_0}}\sqrt{1-\frac{1}{\lvert\mathbf{u}\rvert}}$	2	$\mathbf{E}_2 = (0 \quad 0 \quad 0 \quad u_2 \quad -u_1 \quad 0 \quad -\lambda_2 u_2 \quad \lambda_2 u_1 \quad 0)$; $\mathbf{E}_3 = (0 \quad 0 \quad 0 \quad u_3 \quad 0 \quad -u_1 \quad -\lambda_3 u_3 \quad 0 \quad \lambda_3 u_1)$
$\lambda_4 = \lambda_5 = \lambda_6 = 0$	3	$\mathbf{E}_4 = (1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$; $\mathbf{E}_5 = (0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$; $\mathbf{E}_6 = (0 \quad 0 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0)$
$\lambda_7 = \lambda_8 = \sqrt{\frac{k}{\rho_0}}\sqrt{1-\frac{1}{\lvert\mathbf{u}\rvert}}$	2	$\mathbf{E}_7 = (0 \quad 0 \quad 0 \quad u_2 \quad -u_1 \quad 0 \quad -\lambda_7 u_2 \quad \lambda_7 u_1 \quad 0)$; $\mathbf{E}_8 = (0 \quad 0 \quad 0 \quad u_3 \quad 0 \quad -u_1 \quad -\lambda_8 u_3 \quad 0 \quad \lambda_8 u_1)$
$\lambda_9 = \sqrt{\frac{k}{\rho_0}}$	1	$\mathbf{E}_9 = (0 \quad 0 \quad 0 \quad u_1 \quad u_2 \quad u_3 \quad -\lambda_9 u_1 \quad -\lambda_9 u_2 \quad -\lambda_9 u_3)$

<u>**Remark 3.3**</u>:

3.3.1 – When finite deformation of the string is considered, the constitutive law (Eq. (4)) becomes

$$T = k\log(1+\varepsilon) \qquad \text{and} \qquad T \geq 0\,. \tag{18}$$

In this case, the system has genuinely nonlinear fields (Carasso *et al.*, 1984).

3.3.2 – The eigenvalues $\pm\sqrt{\frac{k}{\rho_0}}$ define two families of straight characteristics.

3.3.3 – For plane problems, $\mathbf{x} = (x_1, x_2)^t \in R^2$, $\lambda_1 = -\sqrt{\frac{k}{\rho_0}}$, $\lambda_2 = -\sqrt{\frac{k}{\rho_0}}\sqrt{1-\frac{1}{\lvert\mathbf{u}\rvert}}$, $\lambda_3 = \lambda_4 = 0$, $\lambda_5 = \sqrt{\frac{k}{\rho_0}}\sqrt{1-\frac{1}{\lvert\mathbf{u}\rvert}}$, $\lambda_6 = \sqrt{\frac{k}{\rho_0}}$.

3.3.4 – For one-dimensional problems, $\mathbf{x} = x_1 \in R$, $\lambda_1 = -\sqrt{\frac{k}{\rho_0}}$, $\lambda_2 = 0$, $\lambda_3 = \sqrt{\frac{k}{\rho_0}}$.

3.3.5 – If, on the one hand, $\mathbf{g}$ does not depend on $\dot{\mathbf{x}}$ and, on the other hand, $\mathbf{f}$ and $\mathbf{g}$ correspond to potentials, then Eq. (14) is Hamiltonien (Marsden *et al.*, 1977)

4. Admissible discontinuities and conserved quantities

We denote by $[\mathbf{w}] = \mathbf{w}(a+) - \mathbf{w}(a-)$ the jump of the quantity $\mathbf{w}$ along a discontinuity having velocity λ. By rewriting Eq. (13) as a system of conservation laws, the conditions of Rankine-Hugoniot (Jeffrey, 1976; Leroux and Schatzman, 1981) show that

$$\lambda[\mathbf{x}] = \mathbf{0} \quad ; \quad \lambda[\mathbf{u}] + [\mathbf{v}] = \mathbf{0} \quad ; \quad \lambda[\mathbf{v}] + \frac{1}{\rho_0}[\mathbf{T}] = \mathbf{0}\,.$$

and we state the result 4.1 below.

Theorem 4.1:

(i) The unitary tangent **t = u**/|**u**| is continuous through the contact discontinuities having velocity $\pm\sqrt{\dfrac{k}{\rho_0}}$ (but the tension T and the velocity **v** may be discontinuous);

(ii) the tension T is continuous through the contact discontinuities propagating with velocity $\pm\sqrt{\dfrac{k}{\rho_0}}\sqrt{1-\dfrac{1}{|\mathbf{u}|}}$ (but the unitary tangent **t = u**/|**u**| and the velocity **v** may be discontinuous) ;

(iii) the unitary tangent **t = u**/|**u**|, the tension T and the velocity **v** are continuous through the stationary discontinuities (thus, the internal efforts $\mathbf{T} = T\mathbf{t}$ and the velocity **v** are continuous) .

Proof: (i) We have $[\mathbf{T}] = -\rho_0\lambda[\mathbf{v}] = \rho_0\lambda^2[\mathbf{u}] = k[\mathbf{u}]$. Thus,

$$\left[k\frac{|\mathbf{u}|-1}{|\mathbf{u}|}\mathbf{u}\right] - k[\mathbf{u}] = k[\mathbf{u}] - k\left[\frac{\mathbf{u}}{|\mathbf{u}|}\right] - k[\mathbf{u}] = -k\left[\frac{\mathbf{u}}{|\mathbf{u}|}\right] = -k[\mathbf{t}] = \mathbf{0}\,.$$

(ii) We have $[\mathbf{T}] = -\rho_0\lambda[\mathbf{v}] = \rho_0\lambda^2[\mathbf{u}] = k\left(1-\dfrac{1}{|\mathbf{u}|}\right)[\mathbf{u}]$. Thus, $|\mathbf{u}|$ has the same value on the both sides:

$$\left[k\frac{|\mathbf{u}|-1}{|\mathbf{u}|}\mathbf{u}\right] = k\left(1-\frac{1}{|\mathbf{u}|}\right)[\mathbf{u}] \Rightarrow \left[\frac{\mathbf{u}}{|\mathbf{u}|}\right] = \frac{[\mathbf{u}]}{|\mathbf{u}|} \Rightarrow [|\mathbf{u}|] = \mathbf{0}\,.$$

(iii) It is immediate that $[\mathbf{T}] = [\mathbf{v}] = 0$.

5. Simple waves and the Riemann Problem.

Let us consider the situation where the external forces **f** and **g** are constant: then the solution is formed of an assembly of constant fields (Jeffrey, 1976). In addition, we set

$$\overline{\mathbf{U}} = (\overline{\mathbf{u}} \quad \overline{\mathbf{v}})^t = (\mathbf{u} \quad \mathbf{v} - \rho_0\mathbf{g}\,t)^t = (u_1, u_2, u_3, v_1 - t\rho_0 g_1, v_2 - t\rho_0 g_2, v_3 - t\rho_0 g_3)^t$$

and we have

$$\overline{\mathbf{U}}_t + \overline{\mathbf{A}}(\overline{\mathbf{U}})\overline{\mathbf{U}}_a = 0 \quad , \quad (0 < a < \ell\,, 0 < t < T)\,, \tag{19}$$

$$\overline{\mathbf{A}}(\overline{\mathbf{U}}) = \begin{pmatrix} 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ \alpha_{11}(\overline{\mathbf{u}}) & \alpha_{12}(\overline{\mathbf{u}}) & \alpha_{13}(\overline{\mathbf{u}}) & 0 & 0 & 0 \\ \alpha_{21}(\overline{\mathbf{u}}) & \alpha_{22}(\overline{\mathbf{u}}) & \alpha_{23}(\overline{\mathbf{u}}) & 0 & 0 & 0 \\ \alpha_{31}(\overline{\mathbf{u}}) & \alpha_{32}(\overline{\mathbf{u}}) & \alpha_{33}(\overline{\mathbf{u}}) & 0 & 0 & 0 \end{pmatrix}\,, \tag{22}$$

The eigenvalues and eigenvectors of $\overline{\mathbf{A}}(\overline{\mathbf{U}})$ are given in Table 2.

Table 2. Eigenvalues of $\overline{\mathbf{A}}(\overline{\mathbf{U}})$.

Eigenvalue	Multiplicity	Example of associated eingenvector
$\overline{\lambda}_1 = -\sqrt{\dfrac{k}{\rho_0}}$	1	$\overline{\mathbf{E}}_1 = \left(\overline{u}_1 \quad \overline{u}_2 \quad \overline{u}_3 \quad -\overline{\lambda}_1\overline{u}_1 \quad -\overline{\lambda}_1\overline{u}_2 \quad -\overline{\lambda}_1\overline{u}_3\right)$
$\overline{\lambda}_2 = \overline{\lambda}_3 = -\sqrt{\dfrac{k}{\rho_0}}\sqrt{1-\dfrac{1}{\lvert\overline{\mathbf{u}}\rvert}}$	2	$\mathbf{E}_2 = \left(\overline{u}_2 \quad -\overline{u}_1 \quad 0 \quad -\lambda_2\overline{u}_2 \quad \lambda_2\overline{u}_1 \quad 0\right)$; $\mathbf{E}_3 = \left(\overline{u}_3 \quad 0 \quad -\overline{u}_1 \quad -\lambda_3\overline{u}_3 \quad 0 \quad \lambda_3\overline{u}_1\right)$
$\overline{\lambda}_4 = \overline{\lambda}_5 = \sqrt{\dfrac{k}{\rho_0}}\sqrt{1-\dfrac{1}{\lvert\overline{\mathbf{u}}\rvert}}$	2	$\overline{\mathbf{E}}_4 = \left(\overline{u}_2 \quad -\overline{u}_1 \quad 0 \quad -\overline{\lambda}_4\overline{u}_2 \quad \overline{\lambda}_4\overline{u}_1 \quad 0\right)$; $\overline{\mathbf{E}}_5 = \left(\overline{u}_3 \quad 0 \quad -\overline{u}_1 \quad -\overline{\lambda}_5\overline{u}_3 \quad 0 \quad \overline{\lambda}_5\overline{u}_1\right)$
$\overline{\lambda}_6 = \sqrt{\dfrac{k}{\rho_0}}$	1	$\overline{\mathbf{E}}_6 = \left(\overline{u}_1 \quad \overline{u}_2 \quad \overline{u}_3 \quad -\overline{\lambda}_6\overline{u}_1 \quad -\overline{\lambda}_6\overline{u}_2 \quad -\overline{\lambda}_6\overline{u}_3\right)$

5.1. Simple waves

The simple waves are obtained by solving the differential equations $d\overline{\mathbf{U}}/d\xi = \overline{\mathbf{E}}_i$. The solutions are given in Table 3. The waves associated to the eigenvalues $\overline{\lambda}_1$ and $\overline{\lambda}_6$ are longitudinal tension ones carrying variations of the tension: the unitary tangent $\overline{\mathbf{t}} = \overline{\mathbf{u}}/\lvert\overline{\mathbf{u}}\rvert$ is constant while $\lvert\overline{\mathbf{u}}\rvert$ varies. The simple waves associated to the eigenvalues $\overline{\lambda}_2$, $\overline{\lambda}_3$, $\overline{\lambda}_4$, $\overline{\lambda}_5$ are deformation ones carrying variations of the unitary tangent $\overline{\mathbf{t}} = \overline{\mathbf{u}}/\lvert\overline{\mathbf{u}}\rvert$ while $\lvert\overline{\mathbf{u}}\rvert$ remains constant.

Table 3. Simple Waves associated to the Eq. (19)

Eigenvalue	Elementary solution
$\overline{\lambda}_1$	$\overline{\mathbf{u}} = \overline{\mathbf{u}}_0 e^{\xi}$; $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \overline{\lambda}_1\left(1-e^{\xi}\right)\overline{\mathbf{u}}_0$
$\overline{\lambda}_2$	$\overline{\mathbf{u}} = \left(A\sin(\xi+\theta_0) \quad A\cos(\xi+\theta_0) \quad B\right)^t$, $\overline{\mathbf{u}}_0 = \left(A\sin(\theta_0) \quad A\cos(\theta_0) \quad B\right)^t$ $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \left(\overline{\lambda}_2 A(\sin(\theta_0)-\sin(\xi+\theta_0)) \quad \overline{\lambda}_2 A(\cos(\theta_0)-\cos(\xi+\theta_0)) \quad 0\right)^t$
$\overline{\lambda}_3$	$\overline{\mathbf{u}} = \left(A\sin(\xi+\theta_0) \quad B \quad A\cos(\xi+\theta_0)\right)^t$, $\overline{\mathbf{u}}_0 = \left(A\sin(\theta_0) \quad B \quad A\cos(\theta_0)\right)^t$ $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \left(\overline{\lambda}_3 A(\sin(\theta_0)-\sin(\xi+\theta_0)) \quad 0 \quad \overline{\lambda}_3 A(\cos(\theta_0)-\cos(\xi+\theta_0))\right)^t$
$\overline{\lambda}_4$	$\overline{\mathbf{u}} = \left(A\sin(\xi+\theta_0) \quad A\cos(\xi+\theta_0) \quad B\right)^t$, $\overline{\mathbf{u}}_0 = \left(A\sin(\theta_0) \quad A\cos(\theta_0) \quad B\right)^t$ $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \left(\overline{\lambda}_4 A(\sin(\theta_0)-\sin(\xi+\theta_0)) \quad \overline{\lambda}_4 A(\cos(\theta_0)-\cos(\xi+\theta_0)) \quad 0\right)^t$
$\overline{\lambda}_5$	$\overline{\mathbf{u}} = \left(A\sin(\xi+\theta_0) \quad B \quad A\cos(\xi+\theta_0)\right)^t$, $\overline{\mathbf{u}}_0 = \left(A\sin(\theta_0) \quad B \quad A\cos(\theta_0)\right)^t$ $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \left(\overline{\lambda}_5 A(\sin(\theta_0)-\sin(\xi+\theta_0)) \quad 0 \quad \overline{\lambda}_5 A(\cos(\theta_0)-\cos(\xi+\theta_0))\right)^t$
$\overline{\lambda}_6$	$\overline{\mathbf{u}} = \overline{\mathbf{u}}_0 e^{\xi}$; $\overline{\mathbf{v}} = \overline{\mathbf{v}}_0 + \overline{\lambda}_6\left(1-e^{\xi}\right)\overline{\mathbf{u}}_0$

5.2. The Riemann Problem

Let us consider the propagation of a discontinuity issued from $a = 0$:

$$\overline{\mathbf{U}} = \overline{\mathbf{U}}_L = \left(\overline{\mathbf{u}}_L \quad \overline{\mathbf{v}}_L\right)^t \ (a < 0) \quad ; \quad \overline{\mathbf{U}} = \overline{\mathbf{U}}_R = \left(\overline{\mathbf{u}}_R \quad \overline{\mathbf{v}}_R\right)^t \ (a > 0) \quad .$$

We look for intermediary states $\overline{\mathbf{U}}_i$ such that $\overline{\mathbf{U}}_L \to \overline{\mathbf{U}}_1 \to \overline{\mathbf{U}}_2 \to \overline{\mathbf{U}}_3 \to \overline{\mathbf{U}}_4 \to \overline{\mathbf{U}}_5 \to \overline{\mathbf{U}}_R$, where each change corresponds to a simple wave associated to $\overline{\lambda}_i$. We observe that the waves corresponding to $\overline{\lambda}_2$ and $\overline{\lambda}_3$ have the same speed: the state $\overline{\mathbf{U}}_2$ cannot be observed. Analogously, $\overline{\mathbf{U}}_4$ cannot be observed.

The intermediary states are determined by using the Rankine-Hugoniot Equations. Let us denote $\alpha = \lambda_6 = \sqrt{k/\rho_0}$. Since $\overline{\mathbf{t}}_1 = \overline{\mathbf{t}}_L$; $\lvert\overline{\mathbf{u}}_2\rvert = \lvert\overline{\mathbf{u}}_1\rvert$; $\lvert\overline{\mathbf{u}}_3\rvert = \lvert\overline{\mathbf{u}}_2\rvert$; $\lvert\overline{\mathbf{u}}_4\rvert = \lvert\overline{\mathbf{u}}_3\rvert$; $\lvert\overline{\mathbf{u}}_5\rvert = \lvert\overline{\mathbf{u}}_4\rvert$; $\overline{\mathbf{t}}_R = \overline{\mathbf{t}}_5$, we have

$$\left.\begin{array}{c}\overline{\mathbf{v}_1}-\overline{\mathbf{v}_L}=\alpha\left(\left|\overline{\mathbf{u}_1}\right|-\left|\overline{\mathbf{u}_L}\right|\right)\overline{\mathbf{t}_L}\ ;\ \overline{\mathbf{v}_2}-\overline{\mathbf{v}_1}=\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_2}-\overline{\mathbf{t}_L}\right);\\ \overline{\mathbf{v}_3}-\overline{\mathbf{v}_2}=\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_3}-\overline{\mathbf{t}_2}\right);\ \overline{\mathbf{v}_4}-\overline{\mathbf{v}_3}=-\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_3}-\overline{\mathbf{t}_2}\right);\\ \overline{\mathbf{v}_5}-\overline{\mathbf{v}_4}=-\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_R}-\overline{\mathbf{t}_4}\right);\ \overline{\mathbf{v}_R}-\overline{\mathbf{v}_5}=-\alpha\left(\left|\overline{\mathbf{u}_R}\right|-\left|\overline{\mathbf{u}_1}\right|\right)\overline{\mathbf{t}_R}\ .\end{array}\right\}\quad(23)$$

By adding all these equations, we obtain the nonlinear system for the unknowns $\overline{\mathbf{t}_3}$ and $\left|\overline{\mathbf{u}_1}\right|$, which reads as

$$2\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\overline{\mathbf{t}_3}+\alpha\left(\left|\overline{\mathbf{u}_1}\right|-\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_L}+\overline{\mathbf{t}_R}\right)=\overline{\mathbf{v}_R}-\overline{\mathbf{v}_L}+\alpha\left(\overline{\mathbf{u}_R}+\overline{\mathbf{u}_L}\right);\ \left|\overline{\mathbf{t}_3}\right|=1\ . \quad (24)$$

By setting $\overline{\mathbf{t}_3}=(\sin\theta_3\quad\cos\theta_3)$ and $r_1=\left|\overline{\mathbf{u}_1}\right|$, Eq. (24) becomes a nonlinear system of two algebraic equations for the two unknowns θ_3 and r_1. The determination of θ_3 and r_1 determines the values of $\overline{\mathbf{t}_3}$ and $\left|\overline{\mathbf{u}_1}\right|$ and the other unknowns $\overline{\mathbf{v}_1}$, $\overline{\mathbf{v}_2}$, $\overline{\mathbf{v}_3}$, $\overline{\mathbf{v}_4}$, $\overline{\mathbf{v}_5}$, $\overline{\mathbf{t}_2}$, $\overline{\mathbf{t}_4}$, $\overline{\mathbf{t}_5}$ are obtained from Eq. (23).

5.3. Two dimensional situations

The eigenvalues and eigenvectors for two dimensional situations are given in Table 4 and the associated simple waves are given in Table 5.

Table 4. Eigenvalues of $\overline{\mathbf{A}}(\overline{\mathbf{U}})$ for two dimensional situations.

Eigenvalue	Example of associated eingenvector
$\overline{\lambda}_1=-\sqrt{\frac{k}{\rho_0}}$	$\overline{\mathbf{E}}_1=\left(\overline{u}_1\quad\overline{u}_2\quad-\overline{\lambda}_1\overline{u}_1\quad-\overline{\lambda}_1\overline{u}_2\right)$
$\overline{\lambda}_2=-\sqrt{\frac{k}{\rho_0}}\sqrt{1-\frac{1}{\lvert\mathbf{u}\rvert}}$	$\mathbf{E}_2=\left(\overline{u}_2\quad-\overline{u}_1\quad-\lambda_2\overline{u}_2\quad\lambda_2\overline{u}_1\right)$;
$\overline{\lambda}_3=\sqrt{\frac{k}{\rho_0}}\sqrt{1-\frac{1}{\lvert\mathbf{u}\rvert}}$	$\overline{\mathbf{E}}_3=\left(\overline{u}_2\quad-\overline{u}_1\quad-\lambda_3\overline{u}_2\quad\lambda_3\overline{u}_1\right)$;
$\overline{\lambda}_4=\sqrt{\frac{k}{\rho_0}}$	$\overline{\mathbf{E}}_4=\left(\overline{u}_1\quad\overline{u}_2\quad-\overline{\lambda}_4\overline{u}_1\quad-\overline{\lambda}_4\overline{u}_2\right)$

Table 5. Simple waves for two dimensional situations

Eigenvalue	Elementary solution
$\overline{\lambda}_1$	$\overline{\mathbf{u}}=\overline{\mathbf{u}}_0e^{\xi}\quad;\quad\overline{\mathbf{v}}=\overline{\mathbf{v}}_0+\overline{\lambda}_1\left(1-e^{\xi}\right)\overline{\mathbf{u}}_0$
$\overline{\lambda}_2$	$\overline{\mathbf{u}}=\left(A\sin(\xi+\theta_0)\quad A\cos(\xi+\theta_0)\right)\quad,\quad\overline{\mathbf{u}}_0=\left(A\sin(\theta_0)\quad A\cos(\theta_0)\right)$ $\overline{\mathbf{v}}=\overline{\mathbf{v}}_0+\left(\overline{\lambda}_2A(\sin(\theta_0)-\sin(\xi+\theta_0))\quad\overline{\lambda}_2A(\cos(\theta_0)-\cos(\xi+\theta_0))\right)^t$
$\overline{\lambda}_3$	$\overline{\mathbf{u}}=\left(A\sin(\xi+\theta_0)\quad A\cos(\xi+\theta_0)\right)\quad,\quad\overline{\mathbf{u}}_0=\left(A\sin(\theta_0)\quad A\cos(\theta_0)\right)$ $\overline{\mathbf{v}}=\overline{\mathbf{v}}_0+\left(\overline{\lambda}_3A(\sin(\theta_0)-\sin(\xi+\theta_0))\quad\overline{\lambda}_3A(\cos(\theta_0)-\cos(\xi+\theta_0))\right)^t$
$\overline{\lambda}_4$	$\overline{\mathbf{u}}=\overline{\mathbf{u}}_0e^{\xi}\quad;\quad\overline{\mathbf{v}}=\overline{\mathbf{v}}_0+\overline{\lambda}_4\left(1-e^{\xi}\right)\overline{\mathbf{u}}_0$

The Riemann problem concerns intermediary states $\overline{\mathbf{U}}_i$ such that $\overline{\mathbf{U}}_L\rightarrow\overline{\mathbf{U}}_1\rightarrow\overline{\mathbf{U}}_2\rightarrow\overline{\mathbf{U}}_3\rightarrow\overline{\mathbf{U}}_R$. It is solved by determining $\overline{\mathbf{t}_2}$ and $\left|\overline{\mathbf{u}_1}\right|$ such that

$$2\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\overline{\mathbf{t}_2}+\alpha\left(\left|\overline{\mathbf{u}_1}\right|-\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_L}+\overline{\mathbf{t}_R}\right)=\overline{\mathbf{v}_R}-\overline{\mathbf{v}_L}+\alpha\left(\overline{\mathbf{u}_R}+\overline{\mathbf{u}_L}\right);\left|\overline{\mathbf{t}_2}\right|=1.$$

and

$$\left.\begin{array}{l}\overline{\mathbf{v}_1}-\overline{\mathbf{v}_L}=\alpha\left(\left|\overline{\mathbf{u}_1}\right|-\left|\overline{\mathbf{u}_L}\right|\right)\overline{\mathbf{t}_L}\,;\overline{\mathbf{v}_2}-\overline{\mathbf{v}_1}=\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_2}-\overline{\mathbf{t}_L}\right)\\ \overline{\mathbf{v}_3}-\overline{\mathbf{v}_2}=-\alpha\left(\sqrt{\left|\overline{\mathbf{u}_1}\right|^2-\left|\overline{\mathbf{u}_1}\right|}\right)\left(\overline{\mathbf{t}_R}-\overline{\mathbf{t}_2}\right);\overline{\mathbf{v}_R}-\overline{\mathbf{v}_3}=-\alpha\left(\left|\overline{\mathbf{u}_R}\right|-\left|\overline{\mathbf{u}_1}\right|\right)\overline{\mathbf{t}_R}\,.\end{array}\right\}$$

A complete analysis of the two dimensional Riemann Problem may be found in Hanche-Olsen *et al.*, 2001.

5.4. Examples of simple solutions

Simple waves may be used in order to construct solutions by using the characteristics of the system: the boundary conditions may be interpreted as waves arriving from infinity. For $0 < t < -\ell/\overline{\lambda_1}$, longitudinal waves travel from $a=\ell$ to $a=0$ with speed $\overline{\lambda_1}$. At time $t = -\ell/\overline{\lambda_1}$, the tension wave is reflected by the boundary $a=0$. For instance, let us consider the two dimensional situation where $k=1$, $\rho_0=1$, $\ell=1$, $\mathbf{g}=0$, $\mathbf{f}=-1$, $\overline{\mathbf{u}}(a,0)=\mathbf{e}_1$, $\overline{\mathbf{v}}(a,0)=\mathbf{0}$.

$$\overline{\mathbf{U}}(a,t)=\begin{cases}(1\quad 0\quad 0\quad 0)^t\ ,\ 0<t<1-a,\\ (-2\quad 0\quad -3\quad 0)^t\ ,\ 1-a<t<1+a,\\ (-5\quad 0\quad 0\quad 0)^t\ ,\ 1+a<t<3-a,\\ (-2\quad 0\quad 3\quad 0)^t\ ,\ 3-a<t<3+a,\\ (1\quad 0\quad 0\quad 0)^t\ ,\ 3+a<t<5-a.\end{cases}$$

Let us consider the same previous data except $\mathbf{f}=1$. In this case, longitudinal waves propagate with speed $\overline{\lambda_1}=-1$, since the unitary tangent is constant. At time $t = 1$, the tension wave is reflected by the boundary $a=0$. The solution is

$$\overline{\mathbf{U}}(a,t)=\begin{cases}(1\quad 0\quad 0\quad 0)^t\ ,\ 0<t<1-a,\\ (2\quad 0\quad 1\quad 0)^t\ ,\ 1-a<t<1+a,\\ (3\quad 0\quad 0\quad 0)^t\ ,\ 1+a<t<3-a,\\ (2\quad 0\quad -1\quad 0)^t\ ,\ 3-a<t<3+a,\\ (1\quad 0\quad 0\quad 0)^t\ ,\ 3+a<t<5-a.\end{cases}$$

6. Numerical Approach

Numerical schemes adapted to this problem may be found in Pego and Serre, 1988; Gilquin, 1989; Gilquin and Serre, 1989. Glimm's and Godunov's approaches have shown to be effective to calculate. We show in Fig. 2, Fig. 3 and Fig. 4 the results furnished by a Godunov based method using the Riemmann solution given in section 5.3. The results concern the point $a = 1/2$.

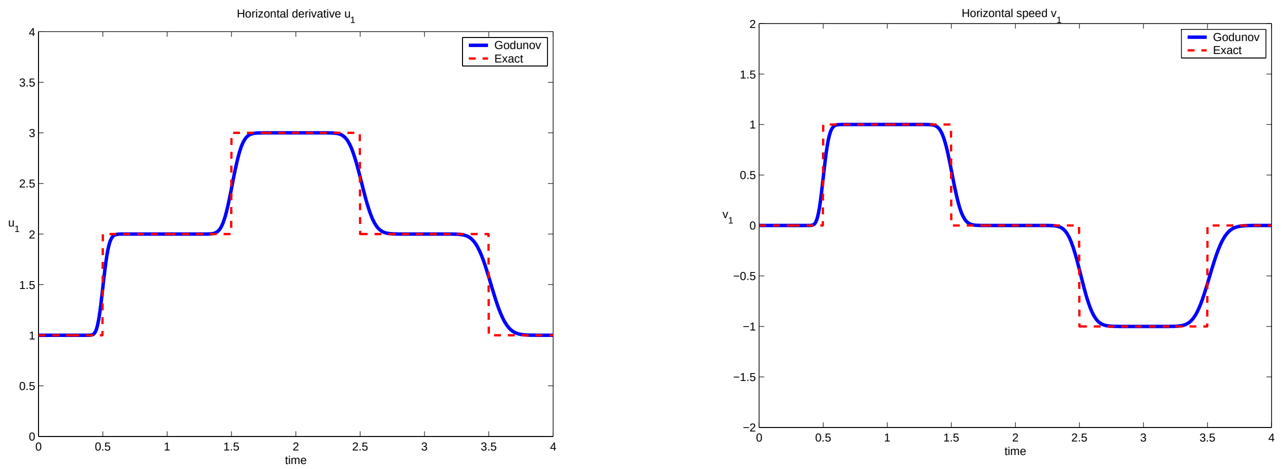

Figure 2 – Numerical results furnished by a Godunov-based approach

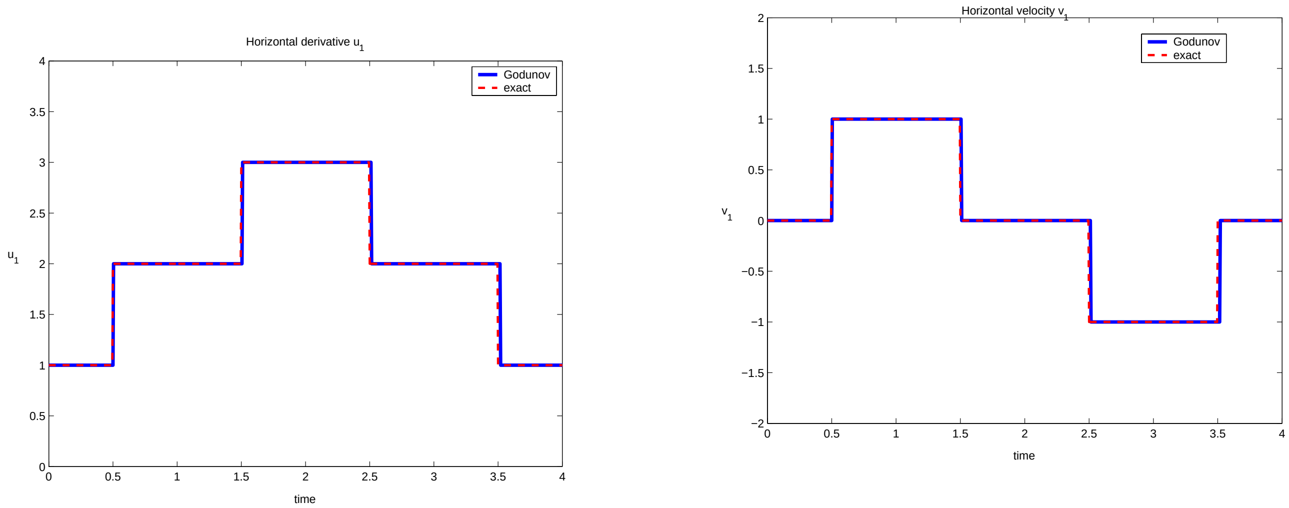

Figure 3 – Numerical results furnished by a Godunov-based approach

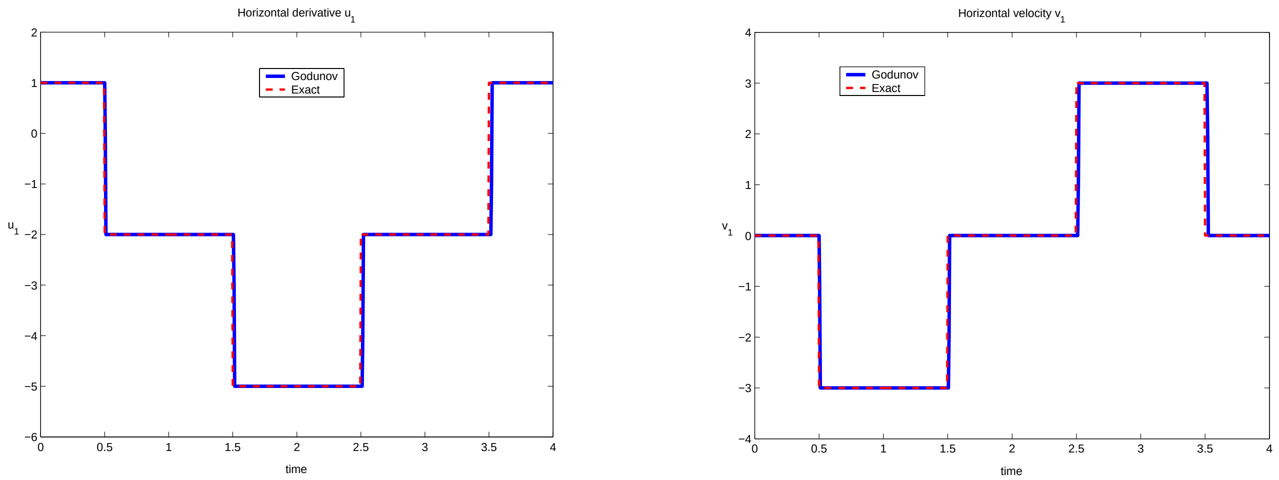

Figure 4 – Numerical results furnished by a Godunov-based approach

7. Concluding Remarks

We have presented a description of strings taking into account their stress unilateral property. The analysis of the equations show that the hyperbolic system associated to the motion of a string is linearly degenerated. The simple waves associated have been obtained and the solution of the Riemann Problem has been presented.

Numerical schemes adapted to this problem may be found the literature. We have presented the results obtained by a Godunov method in some simple situations.

Fabrics and flexible sails are two-dimensional continuous media having the stress unilateral property. The analysis of such a media will be matter of future work.

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9. Responsibility notice

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