Proceedings of the XII International Symposium on Dynamic Problems of Mechanics (DINAME 2007),
P.S.Varoto and M.A.Trindade (editors), ABCM, Ilhabela, SP, Brazil, February 26 - March 2, 2007

Analysis of Nonlinear Systems by Describing Functions - the Experimental Data Issue

Alexandro Garro Brito

Instituto de Aeronáutica e Espaço - Pça. Mal Eduardo Gomes 50, Vl. Acácias. São José dos Campos/SP, Brazil. CEP:12228-904. email: alegbrito@yahoo.com.br

Abstract: This work presents the practical aspects of the use of the describing functions method for nonlinear systems analysis. Numeric and experimental subjects will be discussed. An hydraulic actuator very common in launch vehicle attitude control loop is used as example. The results show that inaccurate conclusions can be produced by restrictive assumptions and problems with acquisition and mathematical procedures.

Keywords: nonlinear systems, describing functions, hydraulic actuators

INTRODUCTION

Physical systems with nonlinear elements can present a very rich dynamical behaviour, including limit-cycle and chaos. In some situations these phenomena are undesirable and it is important adopt strategies to treat them. Limit-cycle is common in attitude control loop of aerospace vehicles. Stout and Snell (2000) presented a study of this phenomenon in on-off aerospace pressure control, while Newman (1995) did a deep discussion of limit-cycles in a launcher control due to nonlinear hydraulic actuator.

The VLS (Brazilian launcher vehicle) uses hydraulic actuators in attitude loop too. This element has a strong nonlinear characteristic since that an high amplitude limit-cycle was noted so in hardware-in-the-loop simulation as in flight data. Because of this an important amount of work is being devoted for the actuator description. In Brito (2006) was discussed a initial characterisation based on classical experimental analysis like *Takens reconstruction* and *Poincaré section*. Bueno (2004) presented the use of the describing functions method for the control loop with hydraulic actuator, however the predictions of the limit-cycle's amplitude and frequency were imprecise. The describing function method is deeply discussed in Sastry (1999) and Slotine and Li (1991).

This work will discuss some details involving the describing functions method and its application in the limit-cycle analysis. As it will be seen, this methodology is very useful to predict the existence of limit-cycle in closed loop as well it can give approximated values for amplitude and frequency of the oscillations. However the predictions can become very inaccurate because of poor data acquisition, very restrictive assumptions and mathematical procedures. Herein these difficulties and possible remedies will be presented. The discussion is based both in open and closed loop nonlinear identification of the hydraulic actuator used in VLS. As conclusion, it will be shown that the describing functions method can not be the best way of studying the limit-cycle, overall when the dominant nonlinearity kind is unknown.

DESCRIBING FUNCTIONS

The describing functions analysis is an extension of the frequency response method for linear systems. Under certain conditions, the describing functions can be used to predict and analyse nonlinear systems, mainly the limit-cycle behaviour in closed loop systems.

The idea is similar to the linear frequency response method. For linear systems, if it is applied a sinusoidal input the output will be a sinusoidal signal with same frequency. However the output is often a periodic, but generally a non-sinusoidal signal when system present some kind of nonlinearity. By using Fourier series, this output can be expanded in a sum of sinusoidal and co-sinusoidal signals in the fundamental frequency (same of the input) and in the higher harmonics. The describing function is the amplitude's and phase's relationships between the input sinusoidal and the fundamental component of the output.

As it is discussed in Slotine and Li (1991), the nonlinear element has to satisfy the following conditions for good results by the describing function analysis:

i) only the fundamental component of the output has to be analysed. This is the most important assumption of the describing function method;

ii) the nonlinear element is time-invariant because the describing functions analysis uses the Nyquist Criterion that is applied only to time-invariant systems;

iii) odd nonlinearity such that positive and negative cycles of the input are equally affected by the nonlinear element.

To obtain a basic version of the describing function method, let us suppose the system in Figure 1a, with a sinusoidal input of amplitude A and frequency ω.

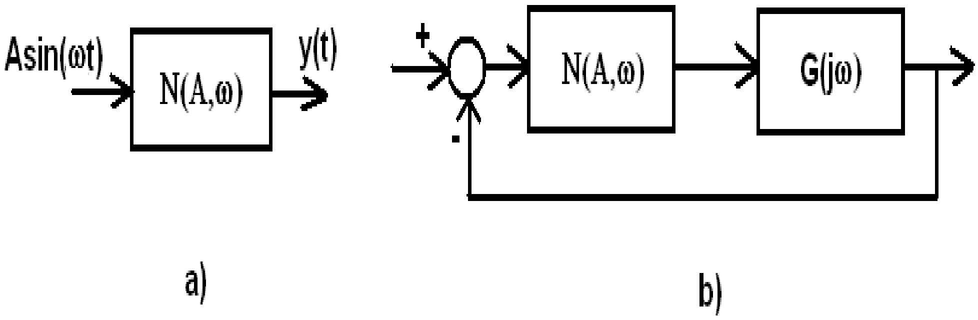

Figure 1 – Diagrams for nonlinear analysis.

The output can be expanded by Fourier series

$$y(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos(n\omega t) + b_n sin(n\omega t), \tag{1}$$

with coefficients given by

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t)\, d(\omega t) \tag{2}$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t) cos(n\omega t)\, d(\omega t) \tag{3}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} y(t) sin(n\omega t)\, d(\omega t). \tag{4}$$

(5)

If the nonlinear element satisfies all conditions of the describing functions method, the coefficient $a_0 = 0$ due to the third assumption, and only the fundamental component has to be considered. Then, the output can be approximated by

$$y(t) \approx a_1 cos(\omega t) + b_1 sin(\omega t) = M\, sin(\omega t + \psi) = \mathscr{I}\{M\, e^{j(\omega t + \psi)}\}, \tag{6}$$

where $\mathscr{I}$ represents the imaginary part of a complex number.

The *describing function* is defined like the complex ratio of the output's fundamental component by the sinusoidal input

$$N(A,\omega) = \frac{M\, e^{j(\omega t + \psi)}}{A\, e^{j\omega t}}. \tag{7}$$

This relationship is dependent of both input frequency and amplitude due to the nonlinearities.

The describing function is used to predict limit-cycle in closed loops with nonlinear elements. In Figure 1b, $G(j\omega)$ represents the linear element in the system while $N(A,\omega)$ is the describing function of the nonlinear element. According to Nyquist criterion, self sustained oscillation occurs in this loop if and only if

$$G(j\omega)N(A,\omega) = -1 \Rightarrow G(j\omega) = \frac{-1}{N(A,\omega)}. \tag{8}$$

Then, the limit-cycle will occur if and only if the curve $G(j\omega)$ intercept the curve $-1/N(A,\omega)$ in the complex plane. The limit-cycle's frequency is given by the value of ω where they intercept themselves. The limit-cycle's amplitude is given by the value of A such that $-1/N(A,\omega) = G(j\omega)$.

HYDRAULIC ACTUATOR

Hydraulic actuators are very common in a lot of processes, overall in aerospace systems; The main manoeuvres of a launcher vehicle are achieved by control systems based in hydraulic actuator commands.

The real behaviour of the hydraulic actuator is quite different of its linear model since that this element has some nonlinearities like *saturation*, *backlash* and *Coulomb and viscous friction* that associates the actuation friction with the main piston speed. These effects seem to be deeply linked with limit-cycle in the aerospace attitude control loop with hydraulic actuators.

A detailed study of the nonlinear phenomena is important for a good prediction of the frequency and amplitude of the limit-cycle. The describing function method is very useful for this, however poor results can be obtained due to the experimental problems and the assumptions required by the method. These questions are discussed below.

ANALYSIS OF THE EXPERIMENTAL DATA

In this section will be presented some results of the hydraulic actuator and the main experimental issues involving the describing function analysis. Two kinds of investigations were done: open loop frequency response and closed loop analysis.

The open loop frequency response is done in the same way of the linear analysis - sinusoidal inputs with different frequencies are applied in the system and its output is studied. The main difference here is that the input's amplitude is varied too, since that both amplitude and frequency can influence the output signal. They were done 91 open loop tests, divided in 7 different amplitudes for the sinusoidal inputs with the following values[1]

$$A_1 = 0.2^\circ \quad A_2 = 0.5^\circ \quad A_3 = 0.7^\circ \quad A_4 = 1.2^\circ \quad A_5 = 1.6^\circ \quad A_6 = 2.1^\circ \quad A_7 = 2.6^\circ \ .$$

For each amplitude above, were created sinusoidal signals with 13 different frequencies (0.1, 0.5, 1, 1.5, 2, 2.5, 3, 4, 5, 7, 10, 15 and 20Hz). The actuator response is presented in the Figure 2. Observe that the nonlinear influence is clear for higher input amplitudes - the amplitude plot has a region in which the values decrease linearly above of a frequency value. It is more difficult conclude something about the phase plot due to the imprecision in the phase measurement. However it is possible note a slight distortion for higher amplitudes and frequencies.

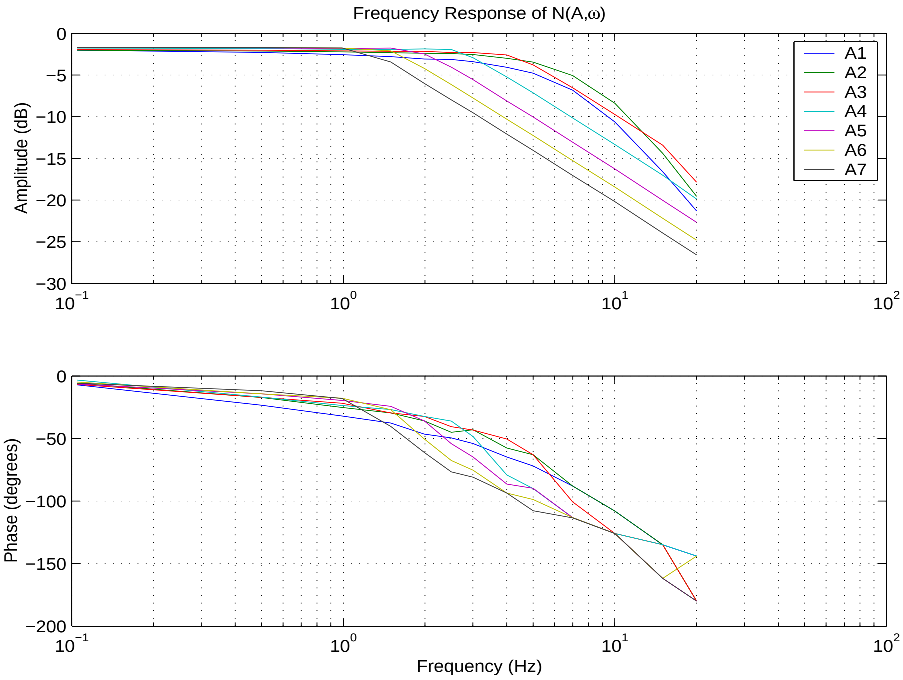

Figure 2 – Frequency response of the hydraulic actuator. Each curve represents one amplitude for the sinusoidal input.

Other tests were performed for the hydraulic actuator in closed loop with a known dynamics. For these experiments, it was used a system with the following representation

$$G(s) = \frac{K_1(K_2 s + 1)}{s^2} \tag{9}$$

where K_1 and K_2 are chosen to produce a limit-cycle ($1 \leq K_1 \leq 100$ and $0.04 \leq K_2 \leq 0.1$). It was tested 30 conditions with

[1] The amplitudes are measured in *degrees* that corresponds to the real deflection of the aerospace movable nozzle that the hydraulic actuator.

K_1 and K_2 in the intervals above; the amplitude and the frequency of the limit-cycle that was produced in each experiment are presented in the Figure 3.

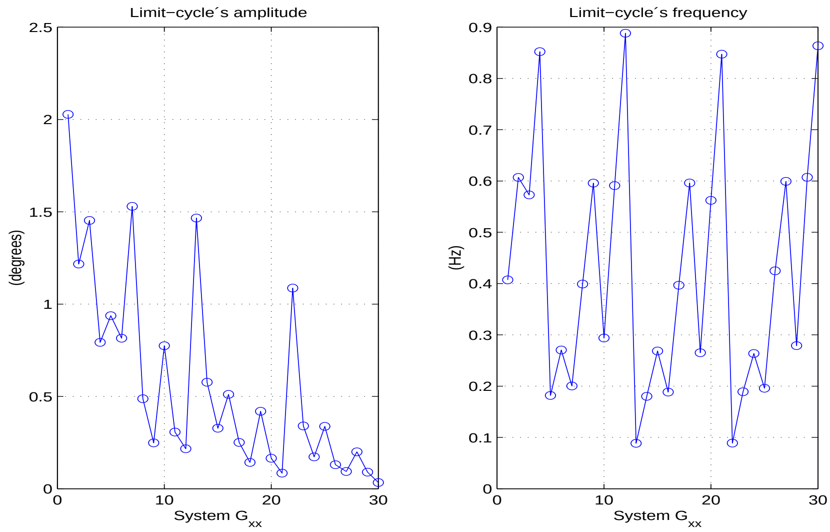

Figure 3 – Frequency and amplitude of the limit cycle for each closed loop experiment.

For applying the describing functions, one should verify the assumptions required by the method. Firstly, the nonlinearities of the actuator must be odd. This implies that the output's positive cycle is equal to the negative cycle. In all tests the hydraulic actuator showed be odd. It must be assured also that the actuator does not have any time-variant behaviour, what was observed in the experimental data.

The most critical assumption is guaranteeing that only the fundamental component of the output spectrum is sufficient to describe it properly. This means that the actuator does not deforms strongly the input signal, preserving approximately the same aspect in the output. One way of testing this condition is measuring the importance of the fundamental component of the output, using the *Total Harmonic Distortion*, defined as

$$\%THD = 100\frac{\sqrt{\sum_{i=2}^{N} y_i^2}}{\sqrt{\sum_{i=1}^{N} y_i^2}}, \tag{10}$$

where y_i is the amplitude of the harmonic component of the output; y_1 is the fundamental component. Low values for THD indicate that the output signal can be well described by its fundamental component. The harmonic distortion analysis for the hydraulic actuator is presented in the Figure 4. Note that the THD is less than 15% in all the tests, indicating that only the fundamental component is suficient for analyze the element behaviour.

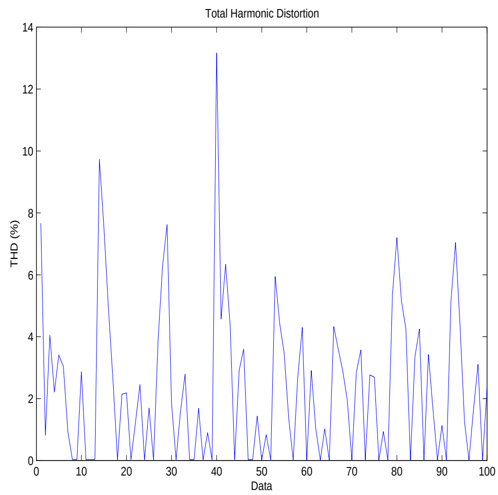

Figure 4 – Total Harmonic Distortion of the hydraulic actuator.

Since that all the necessary assumptions for the describing function method were assured, one can apply it for the analysis of limit cycle in the hydraulic actuator. Initially, it is important predict the limit cycle existence in a closed loop system containing this element. The closed loop exhibit a self-sustained oscillation if the linear's locus crosses the inverse of the nonlinear's locus in the Nyquist plot. The complex representation of the nonlinear element can be extracted of the Figure 2, while the complex representation of the linear part is given in (9). Plotting both complex loci in the same Nyquist plot, one can obtain the Figure 5. Since that there are crossover points between the plots, it is possible predict that the closed loop exhibits a limit-cycle.

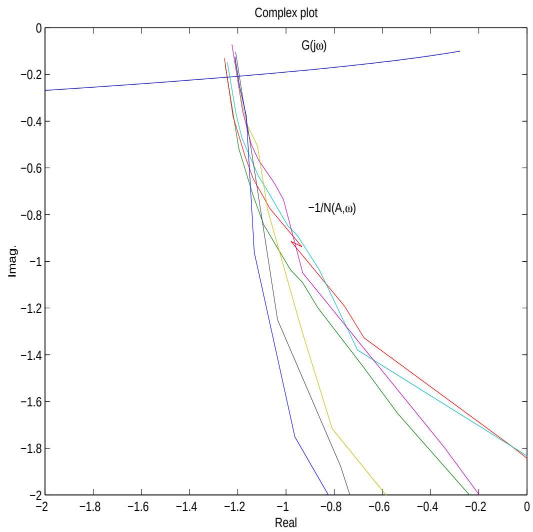

Figure 5 – Nyquist plot of the linear element and the inverse of the nonlinear element. It is possible predict a limit-cycle in a closed loop with both elements.

The next step is trying predict the amplitude and the frequency of the limit-cycle. In the frequency response plot, the prediction of the limit-cycle is slightly similar to those used in the Nyquist plot. A closed loop exhibit limit-cycle if the nonlinear's locus and the inverse of the linear's locus cross themselves at the same frequency in both amplitude and phase plots. So, the limit-cycle frequency will those where the crossover point exists and the limit-cycle amplitude will be those of the respective nonlinear frequency response. The precision of the estimations will depend of the data and mathematical procedures quality. Very inaccurate values can be found due to poor computation of the experimental frequency response. Generally good amplitude estimations are achieved by using FFT algorithms, however the same is not true for the phase. Alternative methods were tested for the phase calculation, including *least squares identification*, but the best results were reached with simple search of the input-output phase delay. Obviously, this is an important source of errors.

To demonstrate how these imprecisions can affect the estimations, let us suppose (9) with $K_1 = 50$ and $K_2 = 0.04$, that corresponds to the system G_2 on the Figure 3. As it can be seen, the amplitude of the limit-cycle is around $1.3°$ while its frequency is $0.6Hz$, both of them obtained in experimental test. By redrawing the frequency response in the Figure 2 adding the response of the inverse of the system G_2, one can get the Figure 6. Note that the amplitude loci cross themselves around $1Hz$ (70% bigger than the real value), but there is any crossover point in the phase plot. The limit-cycle can not be predict by this figure, however it exists in the experimental test.

By using $K_1 = 10$ and $K_2 = 0.06$, that corresponds to the system G_{10}, one can obtain the frequency response presented in Figure 7, with limit-cycle of frequency $0.45Hz$ (assuming a error in the phase estimation). For the same condition, the experimental test exhibits a limit-cycle with frequency of $0.3Hz$, with good agreement. The amplitude of the limit-cycle is given in figure 7 by the A_7 response that is equal to $2.6°$, but the experimental value was of $0.7°$. Again the use of the describing function method failed for the prediction of the limit-cycle measurements.

These results show that the describing function method is not useful for a precise frequency/amplitude prediction, unless a lot of cautions are taken. Initially, it is very important that the nonlinear element attempts the assumptions required by the method. One can obtain incorrect conclusions about the existence of the limit-cycle and its frequency/amplitude values, by applying the analysis to a system in which these requirements are not assured.

Evidently it must be discussed the experimental nature of the describing function method. Errors in the measurements, poor mathematical tools and other sources of inaccuracies can lead the analysis to false conclusions. In this way, other more robust methodologies of nonlinear analysis must be adopted when a better description of the element behaviour is necessary.

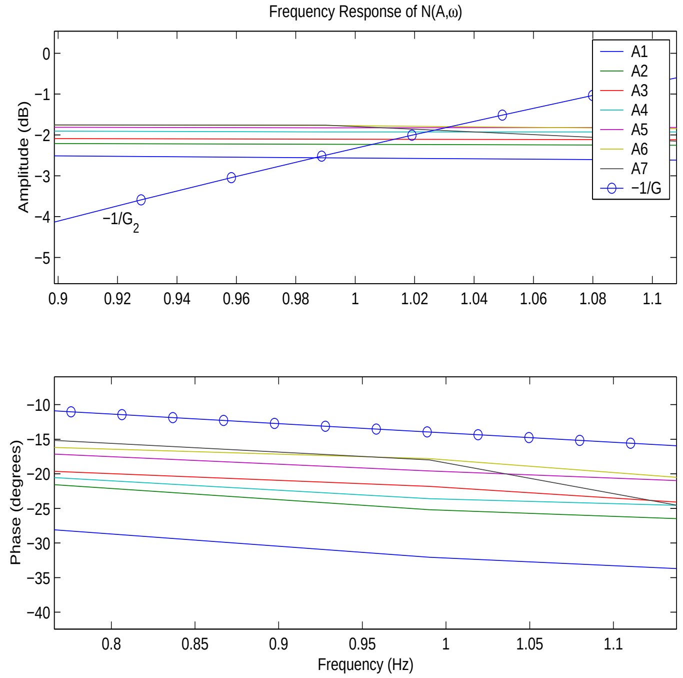

Figure 6 – Frequency response of the nonlinear hydraulic actuator and the inverse of the G_2 system.

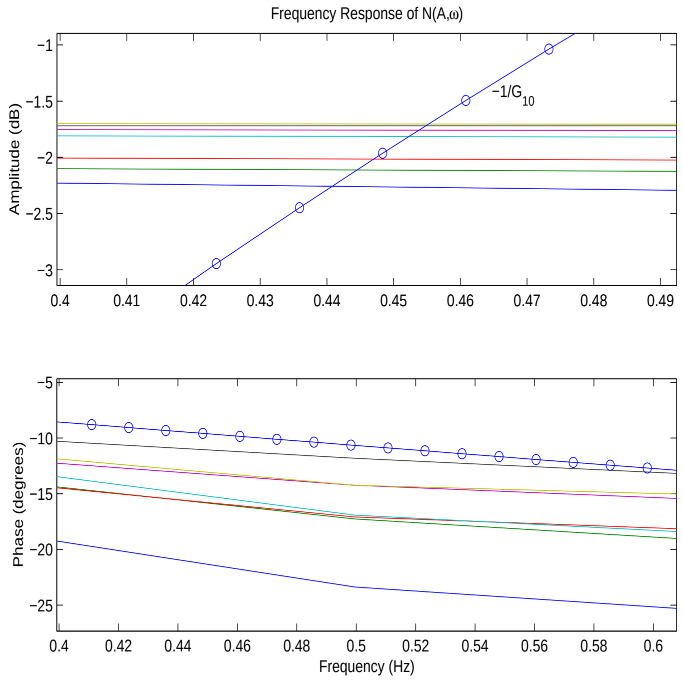

Figure 7 – Frequency response of the nonlinear hydraulic actuator and the inverse of the G_{10} system.

CONCLUSIONS

This work presented experimental issues involving the use of the describing functions method for nonlinear analysis. An hydraulic actuator was used as example since that it can produce limit-cycle when is in closed loop with proper linear dynamics.

As it was discussed, the describing functions method needs a lot of cautions to provide useful informations about limit-cycle in closed loop. Mainly, it is necessary guarantee that the nonlinear element attempts some conditions like odd behaviour and dominance of the fundamental component under the rest harmonics of the output signal. It is important also take care of experimental details, like a good signal acquisition and rich mathematical methods to analyse the data. Without these conditions the describing functions method does note give good results for applications where a closed formulae to nonlinearities are unknown or very difficult to find. In these cases, it is recommended that other methodologies for nonlinear analysis are adopted.

ACKNOWLEDGEMENTS

I would like thank Dr. Waldemar Castro Leite Filho IAE/CTA by the very useful informations about describing functions and nonlinear systems.

REFERENCES

Bueno, A.M., 2004, “Projeto de Sistemas de Controle com Não-linearidades Causadoras de Ciclo-limite”, Master Thesis, Instituto Nacional de Pesquisas Espaciais, São José dos Campos, Brazil.

Brito, A.G., 2006, “Caracterização Não-linear de Sistema de Controle de Atitude com Atuadores de Tubeira Móvel”, IV Congresso Nacional de Engenharia Mecânica, Recife, Brazil.

Newman, B., 1995, “Dynamics and Control of Limit Cycling Motions in Boosting Rockets”, J. of Guidance, Control and Dynamics, Vol. 18, No. 2, pp. 280-286.

Sastry, S., 1999, “Nonlinear Systems - Analysis, Stability and Control”, Ed. Springer, Pasadena, USA, 667 p.

Slotine, J., Li, W. 1991, “Applied Nonlinear Control”, Ed. Prentice Hall, New Jersey, USA, 459 p. pp. 149-167.

Stout, P.W., Snell, S.A., 2000, “Multiple On-Off Valve Control for a Launch Vehicle Tank Pressurization System”, J. of Guidance, Control and Dynamics, Vol. 23, No. 4, pp. 611-619.

RESPONSIBILITY NOTICE