FLOW OF A NON-NEWTONIAN FLUID THROUGH CHANNELS WITH PERMEABLE WALL

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Abstract. *In the present work the momentum transport in two adjacent flow regions is described by means of a continuum theory of mixtures, specially developed to model multiphase phenomena. A generalized Newtonian fluid flows through the permeable wall channel, originating a pure fluid region and a mixture region - where the fluid saturates the porous matrix. The fluid and the porous matrix are treated as continuous constituents of a binary mixture coexisting superposed, each of them occupying simultaneously the whole volume of the mixture. An Ostwald-de Waele behavior is assumed for both the fluid constituent (in the mixture region) and the fluid (in the so-called pure fluid region), while the porous matrix, represented by the solid constituent, is assumed rigid, homogeneous, isotropic and at rest. Compatibility conditions at the interface (pure fluid-mixture) for momentum transfer are proposed and discussed. Assuming no flow across the interface, the velocity should be zero on the solid parts of the boundary and should match the fluid diffusing velocity on the fluid parts of the boundary. Also the shear stress at the pure fluid region is to be balanced by a multiple of the partial shear stress at the mixture region. A minimum principle for the above-described problem, assuming fully developed flow in both regions, is presented, providing an easy and reliable way for carrying out numerical simulations.*

Keywords **:** *Mixture theory, Non-Newtonian fluid, Permeable wall, Minimum Principle.*

1. INTRODUCTION

The present work studies the momentum transport in the flow of a Stokesian fluid through a channel bounded by a permeable wall. Two distinct flow regions are considered in the mathematical model, one is occupied by a Non-Newtonian incompressible fluid (the so-called pure fluid region) while in the other one a porous medium is saturated by the above-mentioned fluid. A continuum theory of mixtures approach - a generalization of the continuum mechanics, specially developed to deal with multiphase phenomena - is employed in the description, in such a way that the region with the porous matrix is denoted by mixture region (a binary solid-fluid mixture is considered) and in the pure fluid the continuum mechanics balance equations are recovered. The mixture region consists of two overlapping continuous constituents: an incompressible fluid constituent, representing the Non-Newtonian fluid and a rigid, homogeneous, isotropic and at rest solid constituent, representing the porous matrix.

Some examples of the above described flows may be found in porous bearing lubrication, flow of perforation mud in oil whells and packed-bed heat exchangers (in which the porous matrix is only present in a vicinity of the hot-cold fluid interface, where the heat exchange is higher).

Transport in porous media are generally modeled by employing a local volume-averaging technique, discussed in detail by Whitaker (1969), to describe quantities such as temperature, pressure, concentration and the velocity components; allowing the use of the classical continuum mechanics approach. Vafai and Kim (1990) have used this approach and Darcy's law - with the addition of empirically determined terms (Brinkmann and Forchheimer extensions) to account for inertia and viscous effects and to satisfy the no-slip condition - as the balance of linear momentum. They analyzed convective flow and heat transfer in two distinct flow regions (fluid and fluid-saturated porous medium) aiming at a fundamental investigation of the interaction phenomena at the interface, where continuity of velocities, pressure, deviatoric normal and shear stresses, temperature and heat flux were imposed. Assuming steady-state flow and local thermal equilibrium, they simulated a problem by means of a control volume method, studying the effects of Darcy and Prandtl numbers, of an inertia parameter (related to the Forchheimer term) and of a conductivity ratio, which relates the porous medium effective conductivity to the fluid conductivity. Huang and Vafai (1993) have studied the forced convection over a complex geometry consisting of multiple porous blocks attached to an impermeable wall. Two distinct flow regions are considered in this arrangement used for flow and heat transfer control: the fluid and the fluid flowing through the porous blocks. A numerical investigation of the flow field and thermal characteristics, using a control volume method, was performed.

A distinct approach is employed in this work to describe the flow of a Non-Newtonian fluid through a channel with permeable wall: the mixture theory, which generates a thermodynamically consistent local model. On the other hand, it gives rise to a momentum source (absent in continuum mechanics) to provide dynamical interaction among the mixture constituents, since the theory allows the existence of as many velocity fields as the number of constituents in the mixture. This approach has already been successfully employed in the modeling of the convection of a Newtonian fluid in two distinct flow regions (Martins-Costa and Saldanha da Gama, 1994). Constitutive relations for the partial stress tensor (Martins-Costa et al., 1992b) and the momentum source (Saldanha da Gama and Sampaio, 1985) for a Stokesian fluid, satisfying the

Second Law of Thermodynamics (Costa Mattos et al., 1993; 1995), have been used.

Compatibility conditions at the interface (pure fluid-mixture) for momentum transfer are proposed and discussed. Kinematics and dynamic compatibility conditions characterize this interface. Considering continuity in the velocity field the velocity should be zero on the solid parts of the boundary (since the porous matrix is supposed to be at rest) and should match the fluid diffusive velocity on the fluid parts of the boundary. It is also expected that both solid and fluid constituents receive shear stress from the fluid stream at the pure fluid region. In other words, the pure fluid shear stress is balanced by a shear stress multiple in the mixture region. It should be noticed that, at the interface, the pure fluid velocity is distinct from the fluid constituent velocity (in the mixture region), when a mixture theory viewpoint is considered.

The mathematical description of this two-region isothermal flow, namely mass and linear momentum conservation for the fluid in the pure fluid region and for the fluid constituent in the mixture region, is simplified by assuming fully developed flow in both regions. A variational formulation has been derived, since this problem has an equivalent minimum principle. The existence of a functional whose minimization is equivalent to the solution of the differential problem is a powerful tool for obtaining numerical approximations.

2. MATHEMATICAL MODELING

In what follows a binary mixture is considered, representing the saturated flow of a generalized Newtonian incompressible fluid through a rigid homogeneous and isotropic porous matrix, supposed to be at rest. Since the solid constituent is assumed rigid, it suffices to solve mass and linear momentum balance equations for the fluid constituent of the mixture.

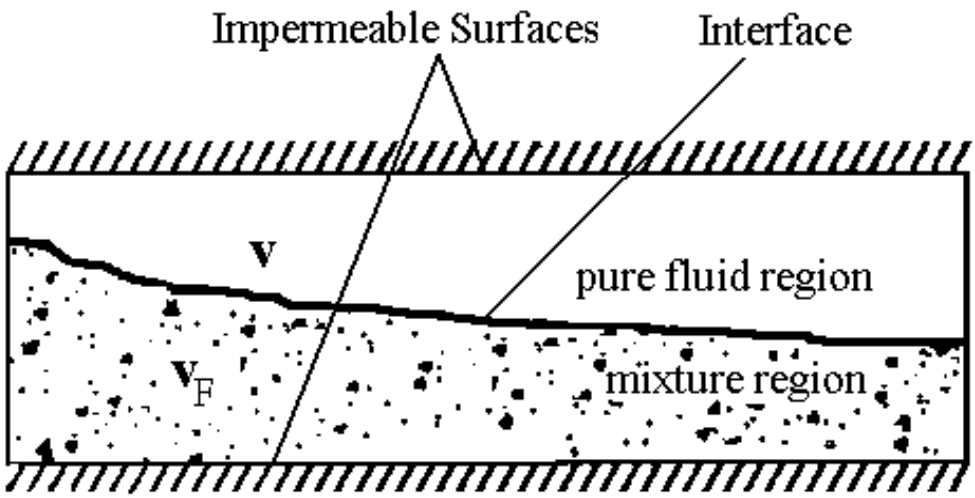

Figure 1 - Typical two-region arrangement.

Representing by the open set Ω_1, with boundary $\partial\Omega_1$, the region occupied by the pure fluid and by the open set Ω_2, with boundary $\partial\Omega_2$, the region occupied by the mixture - as shown in Fig. 1, the mass balances are (Martins-Costa et al., 1992a; Saldanha da Gama and Sampaio, 1983)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 \qquad \text{in } \Omega_1 \tag{1}$$

$$\frac{\partial \rho_{\mathrm{F}}}{\partial t} + \nabla \cdot (\rho_{\mathrm{F}} \mathbf{v}_{\mathrm{F}}) = 0 \qquad \text{in } \Omega_2 \tag{2}$$

in which ρ is the actual fluid density in Ω_1, $\mathbf{v}$ is the fluid velocity in Ω_1, ρ_{F} is the fluid constituent mass density in Ω_2 and $\mathbf{v}_{\mathrm{F}}$ is the fluid constituent velocity in Ω_2. The field ρ_{F} is locally defined as the ratio between the fluid constituent mass and the volume of the mixture. Since a saturated flow is being considered, $\rho_{\mathrm{F}} = \rho\varphi$, where φ represents the porosity. The mass of each constituent is preserved, assuring that the mass of the mixture as a whole is automatically conserved, in the absence of chemical reactions.

The balance of linear momentum is given by (Martins-Costa et al., 1992a; Saldanha da Gama and Sampaio, 1983):

$$\rho[\frac{\partial \mathbf{v}}{\partial t} + (\nabla \mathbf{v})\mathbf{v}] = \nabla \cdot \boldsymbol{\sigma} + \rho \mathbf{g} \qquad \text{in } \Omega_1 \tag{3}$$

$$\rho_{\mathrm{F}}[\frac{\partial \mathbf{v}_{\mathrm{F}}}{\partial t} + (\nabla \mathbf{v}_{\mathrm{F}})\mathbf{v}_{\mathrm{F}}] = \nabla \cdot \boldsymbol{\sigma}_{\mathrm{F}} + \mathbf{m}_{\mathrm{F}} + \rho_{\mathrm{F}} \mathbf{g} \qquad \text{in } \Omega_2 \tag{4}$$

in which $\boldsymbol{\sigma}$ is the Cauchy stress tensor, $\boldsymbol{\sigma}_{\mathrm{F}}$ is the partial stress tensor associated to the fluid constituent, $\mathbf{g}$ is a body force per unit mass and $\mathbf{m}_{\mathrm{F}}$ is an interaction force per unit volume acting on the fluid constituent, due to its interaction with the solid constituent of the binary mixture. The field $\mathbf{m}_{\mathrm{F}}$ is a momentum source, arising from the existence of more than one velocity at each point in the mixture region. Since $\mathbf{m}_{\mathrm{F}}$ is an internal contribution, considering a solid-fluid mixture, the relation $\mathbf{m}_{\mathrm{F}} = -\mathbf{m}_{\mathrm{S}}$ must hold everywhere, with $\mathbf{m}_{\mathrm{S}}$ denoting the interaction force acting on the solid constituent.

2.1 Constitutive Assumptions

Non-Newtonian behavior is characterized by the shear stress being no longer linearly proportional to the velocity gradients (as in the Newtonian case). Among these fluids three broad groups may be distinguished: purely viscous, viscoelastic and time-dependent fluids (Cho and Hartnett, 1982). Purely viscous fluids are also called generalized Newtonian fluids because their models may be viewed as modifications of Newton's law of viscosity to allow the viscosity to become a function of the shear rate. Generalized Newtonian models include, among others, the Ostwald-de Waele fluid (also known as power-law fluid), the Bingham plastic, the Ellis fluid and the Eyring fluid. The power-law model, despite its simplicity, is a good approximation for most Non-Newtonian fluids, particularly in the laminar flow regime. This simplification may even include pipe flow of viscoelastic fluids, resulting from the fact that their elastic nature does not play a significant role in laminar pipe flow (Cho and Hartnett, 1982).

In this work an Ostwald-de Waele model is employed to describe the fluid flow in the region $\Omega_1 \cup \Omega_2$. The following constitutive equation for the stress tensor is employed in the pure fluid region (Bird et al., 1977):

$$\boldsymbol{\sigma} = -p\mathbf{1} + 2\eta_0 (\mathbf{D} \cdot \mathbf{D})^n \mathbf{D} \qquad \text{in } \Omega_1 \tag{5}$$

where p is the pressure acting on the fluid and η_0 and n are the Ostwald-de Waele parameters of the fluid. Equation (5), also known as power-law, is reduced to Newton's law of viscosity if $n = 0$ and η_0 is the fluid viscosity. The deviation of Newtonian behavior is related to the deviation of n from zero. Power-law fluids may be classified as shear thickening and shear-thinning according to the value of n. They have dilatant behavior when $n > 0$, being pseudo-plastic for $n < 0$.

An analogy with the partial stress tensor proposed by Williams (1978) for a Newtonian fluid leads to the following constitutive relation for the partial stress tensor acting on the generalized Newtonian fluid described by Eq. (5) (Martins-Costa et al., 1992b)

$$\boldsymbol{\sigma}_{\mathrm{F}} = -\varphi p\mathbf{1} + 2\lambda\varphi^2\eta_0(\mathbf{D}_{\mathrm{F}} \cdot \mathbf{D}_{\mathrm{F}})^n\mathbf{D}_{\mathrm{F}} \qquad \text{in } \Omega_2 \tag{6}$$

where λ is a scalar parameter depending on the porous matrix microstructure and $\mathbf{D}_{\mathrm{F}}$ is the symmetric part of the $\nabla\mathbf{v}_{\mathrm{F}}$.

According to Saldanha da Gama and Sampaio (1985) the interaction force acting on the fluid constituent, considering an Ostwald-de Waele behavior described by Eq. (5) for the fluid as a continuum, is given by

$$\mathbf{m}_{\mathrm{F}} = -\frac{\varphi^2\eta}{K}\Big[\Big(\frac{4n+3}{2n+1}\Big)^{2n+1}\frac{1}{3}\Big(\frac{\varphi}{6K}\Big)^n\|\mathbf{v}_{\mathrm{F}}\|^{2n}\Big]\mathbf{v}_{\mathrm{F}} \qquad \text{in } \Omega_2 \tag{7}$$

in which K is the porous matrix specific permeability and the solid constituent has been assumed at rest.

2.2 Problem Description

The problem represented by equations (1)-(4), combined with constitutive assumptions (5)-(7), assuming a steady-state fully developed flow of an Ostwald-de Waele incompressible fluid flowing in both regions, may be stated as

$$\begin{aligned}
\nabla \cdot \mathbf{v} &= 0 && \text{in } \Omega_1 \\
\nabla \cdot \mathbf{v}_{\mathrm{F}} &= 0 && \text{in } \Omega_2 \\
\nabla \cdot \Big[-p\mathbf{1} + 2\eta_0(\mathbf{D} \cdot \mathbf{D})^n\mathbf{D}\Big] + \rho\mathbf{g} &= 0 && \text{in } \Omega_1 \\
\nabla \cdot \Big[-\varphi p\mathbf{1} + 2\lambda\varphi^2\eta_0(\mathbf{D}_{\mathrm{F}} \cdot \mathbf{D}_{\mathrm{F}})^n\mathbf{D}_{\mathrm{F}}\Big] + \gamma_{\mathrm{F}}\|\mathbf{v}_{\mathrm{F}}\|^{2n}\mathbf{v}_{\mathrm{F}} + \rho_{\mathrm{F}}\mathbf{g} &= 0 && \text{in } \Omega_2 \\
\gamma_{\mathrm{F}} = -\frac{\varphi^2\eta}{K}\Big[\Big(\frac{4n+3}{2n+1}\Big)^{2n+1}\frac{1}{3}\Big(\frac{\varphi}{6K}\Big)^n &&
\end{aligned} \tag{8}$$

Equation (8) requires, besides the classical no-slip conditions ($\mathbf{v} = 0$ and $\mathbf{v}_{\mathrm{F}} = 0$) at the impermeable boundaries, compatibility conditions at the interface pure fluid-mixture. The interface between the regions Ω_1 and Ω_2 is defined by the set $\partial\Omega_I \equiv \overline{\Omega}_1 \cap \overline{\Omega}_2$. At this interface some compatibility conditions must be imposed in order to allow the solution of the problem. According to Williams (1978), since there is no flow across the interface, the following relations must hold:

$$\begin{aligned} \mathbf{v}\cdot\mathbf{n} = \mathbf{v}_{\mathrm{F}}\cdot\mathbf{n} = 0 \qquad & \text{on} \quad \partial\Omega_I \\ \mathbf{v} = \varphi\mathbf{v}_{\mathrm{F}} \qquad & \text{on} \quad \partial\Omega_I \\ \varphi\boldsymbol{\sigma}\mathbf{n}\cdot\mathbf{t} = \boldsymbol{\sigma}_{\mathrm{F}}\mathbf{n}\cdot\mathbf{t} \qquad & \text{on} \quad \partial\Omega_I \end{aligned} \tag{9}$$

in which $\mathbf{n}$ is an outward normal to $\partial\Omega_I$ and $\mathbf{t}$ is any tangent to $\partial\Omega_I$. The compatibility equations (9) simulate the experimental condition proposed by Beavers and Joseph (1967), which was confirmed and generalized by several authors (Nield and Bejan, 1992). The idealized model for a porous medium, employed by Taylor (1971) and Richardson (1971) gives theoretical support to Beavers and Joseph's condition. Equations (9), obtained from the solution of thermodynamically consistent equations (derived by means of a mixture theory viewpoint) in both regions, do not suffer from the difficulty of matching the porous medium flow equations with the fluid equation, discussed by Nield and Bejan (1992). Williams (1978), based on a no-slip condition, concluded that the velocity should be zero on the solid parts of the boundary (since the porous matrix is at rest) and should match the fluid diffusing velocity on the fluid parts of the boundary. He also supposed that both solid and fluid receive shearing stress from the fluid stream at the pure fluid region. It should be noticed that, at the interface, the pure fluid velocity is distinct from the fluid constituent velocity (in the mixture region), when a mixture theory viewpoint is considered.

3. THE MINIMUM PRINCIPLE

The problem described by Eqs. (8)-(9) is equivalent to the minimization the following functional (Kreyszig, 1978)

$$\begin{aligned} \Pi = & -\int_{\Omega_1} 2\eta_0 \, \frac{\|\mathbf{D}\|^{2n+2}}{2n+2} \, dV + \int_{\Omega_1} (\rho\mathbf{g} - \nabla p)\cdot\mathbf{v} \, dV \\ & - \int_{\Omega_2} 2\lambda\varphi^2\eta_0 \, \frac{\|\mathbf{D}_{\mathrm{F}}\|^{2n+2}}{2n+2} \, dV - \int_{\Omega_2} \gamma_{\mathrm{F}} \, \frac{\|\mathbf{v}_{\mathrm{F}}\|^{2n+2}}{2n+2} \, dV + \int_{\Omega_2} (\rho\varphi\mathbf{g} - \varphi\nabla p)\cdot\mathbf{v}_{\mathrm{F}} \, dV \end{aligned} \tag{10}$$

where $\|*\|$ corresponds to the Euclidean norm $(*\cdot *)^{1/2}$.

The minimization of the functional Π in Eq. (10) is considered for a space of admissible fields $\{\mathbf{u}\}$ (Mikhin, 1964) which must be such that the essential conditions

$$\begin{aligned} \nabla\cdot\mathbf{v} = 0 \quad & \text{and} \quad \nabla\cdot\mathbf{v}_{\mathrm{F}} = 0 & \\ \mathbf{v}\cdot\mathbf{n} = 0 \quad & \text{and} \quad \mathbf{v}_{\mathrm{F}}\cdot\mathbf{n} = 0 \qquad & \text{on} \quad \partial\Omega_I \\ \mathbf{v} = 0 \quad & \text{and} \quad \mathbf{v}_{\mathrm{F}} = 0 \qquad & \text{on} \quad \partial\Omega_{\mathrm{imp}} \\ \mathbf{v}\cdot\mathbf{t} = \varphi\mathbf{v}_{\mathrm{F}}\cdot\mathbf{t} \qquad & & \text{on} \quad \partial\Omega_I \end{aligned} \tag{11}$$

are satisfied. The first two equations in Eq. (11) characterize the admissible functions and $\partial\Omega_{\mathrm{imp}}$ denote any impermeable boundary.

Taking the first variation of Π (Mikhin, 1964),

$$
\begin{aligned}
\delta\Pi = &- \int_{\Omega_1} 2\eta_0 \ \|\mathbf{D}\|^{2n+1} \times \delta\|\mathbf{D}\| \ dV + \int_{\Omega_1} (\rho\mathbf{g} - \nabla p) \cdot \delta\mathbf{v} \ dV \\
&- \int_{\Omega_2} 2\lambda\varphi^2\eta_0 \ \|\mathbf{D}_\mathrm{F}\|^{2n+1} \times \delta\|\mathbf{D}_\mathrm{F}\| \ dV - \int_{\Omega_2} \gamma_\mathrm{F} \ \|\mathbf{v}_\mathrm{F}\|^{2n+1} \cdot \delta\|\mathbf{v}_\mathrm{F}\| \ dV \\
&+ \int_{\Omega_2} (\rho\varphi\mathbf{g} - \varphi\nabla p) \cdot \delta\mathbf{v}_\mathrm{F} \ dV
\end{aligned}
\tag{12}
$$

Now, applying the Divergence Theorem and imposing the first variation of Π to be zero, the following relation is obtained

$$
\begin{aligned}
&\int_{\Omega_1} \left\{ 2\eta_0 \ \nabla \cdot \left[\|\mathbf{D}\|^{2n}\mathbf{D} \right] \cdot \delta\mathbf{v} + (\rho\mathbf{g} - \nabla p) \cdot \delta\mathbf{v} \right\} dV - \int_{\partial\Omega_1} \eta_0 \|\mathbf{D}\|^{2n}\mathbf{D} \ \mathbf{n}_1 \cdot \delta\mathbf{v} \ dS \\
&+ \int_{\Omega_2} \left\{ 2\lambda\varphi^2\eta_0 \ \nabla \cdot \left[\|\mathbf{D}_\mathrm{F}\|^{2n}\mathbf{D}_\mathrm{F} \right] \cdot \delta\mathbf{v}_\mathrm{F} + \varphi(\rho\mathbf{g} - \nabla p) \cdot \delta\mathbf{v}_\mathrm{F} - \gamma_\mathrm{F} \ \|\mathbf{v}_\mathrm{F}\|^{2n}\mathbf{v}_\mathrm{F} \cdot \delta\mathbf{v}_\mathrm{F} \right\} dV \\
&- \int_{\partial\Omega_2} 2\lambda\varphi^2\eta_0 \|\mathbf{D}_\mathrm{F}\|^{2n}\mathbf{D}_\mathrm{F} \ \mathbf{n}_2 \cdot \delta\mathbf{v}_\mathrm{F} \ dS \ = \ 0
\end{aligned}
\tag{13}
$$

where $\mathbf{n}_i$ represents the unit outward normal to the surface $\partial\Omega_i$.

At this point one can conclude that the Euler-Lagrange equations corresponding to the variational principle stated in Eq. (13) are the equation of continuity and the components of the equation of motion. Considering an Ostwald-de Waele fluid in a steady-state fully developed flow in a channel with permeable wall, they are represented by Eq. (8). Besides, the natural boundary conditions are given by

$$
\int_{\partial\Omega_1} \eta_0 \|\mathbf{D}\|^{2n}\mathbf{D} \ \mathbf{n}_1 \cdot \delta\mathbf{v} \ dS + \int_{\partial\Omega_2} 2\lambda\varphi^2\eta_0 \|\mathbf{D}_\mathrm{F}\|^{2n}\mathbf{D}_\mathrm{F} \ \mathbf{n}_2 \cdot \delta\mathbf{v}_\mathrm{F} \ dS \ = \ 0 \tag{14}
$$

It can be shown that the boundary conditions stated in Eq. (14) are equivalent to Eq. (9c), which may be stated as

$$
\eta_0 \|\mathbf{D}\|^{2n}\mathbf{D} \ \mathbf{n} \cdot \mathbf{t} = \lambda\varphi^2\eta_0 \|\mathbf{D}_\mathrm{F}\|^{2n}\mathbf{D}_\mathrm{F} \ \mathbf{n} \cdot \mathbf{t} \qquad \text{on} \ \ \partial\Omega_I \tag{15}
$$

The boundaries $\partial\Omega_1$ and $\partial\Omega_2$ may either be composed by a permeable surface separating the pure-fluid and the mixture regions or by impermeable surfaces, so Eq. (14) may be rewritten as

$$
\begin{aligned}
&\int_{\partial\Omega_1 \cap \partial\Omega_2} \eta_0 \|\mathbf{D}\|^{2n}\mathbf{D} \ \mathbf{n}_1 \cdot \delta\mathbf{v} + 2\lambda\varphi^2\eta_0 \|\mathbf{D}_\mathrm{F}\|^{2n}\mathbf{D}_\mathrm{F} \ \mathbf{n}_2 \cdot \delta\mathbf{v}_\mathrm{F} \ dS \\
&+ \int_{\partial\Omega_1 \cap \partial\Omega_{\mathrm{imp}}} \eta_0 \|\mathbf{D}\|^{2n}\mathbf{D} \ \mathbf{n}_1 \cdot \delta\mathbf{v} \ dS + \int_{\partial\Omega_2 \cap \partial\Omega_{\mathrm{imp}}} 2\lambda\varphi^2\eta_0 \|\mathbf{D}_\mathrm{F}\|^{2n}\mathbf{D}_\mathrm{F} \ \mathbf{n}_2 \cdot \delta\mathbf{v}_\mathrm{F} \ dS \ = \ 0
\end{aligned}
\tag{16}
$$

Since on the impermeable boundaries $\partial\Omega_{\text{imp}}$, both fields $\mathbf{v}$ and $\mathbf{v}_{\text{F}}$ are prescribed, the second and the third integrals of Eq. (16) are zero everywhere. In the first integral, since $\partial\Omega_1$ and $\partial\Omega_2$ are adjacent regions, $\partial\Omega_1 \cap \partial\Omega_2 \neq \emptyset$. So, Eq. (16) is satisfied if

$$\eta_0 \|\mathbf{D}\|^{2n} \mathbf{D}\ \mathbf{n}_1 \cdot \delta\mathbf{v} + 2\lambda\varphi^2 \eta_0 \|\mathbf{D}_{\text{F}}\|^{2n} \mathbf{D}_{\text{F}}\ \mathbf{n}_2 \cdot \delta\mathbf{v}_{\text{F}} \ = \ 0 \tag{17}$$

Since $\partial\Omega_1 \cap \partial\Omega_2$ is the interface separating the adjacent regions, it comes that

$$\mathbf{n}_1 = -\mathbf{n}_2 = \mathbf{n} \qquad\qquad \text{on} \quad \partial\Omega_1 \cap \partial\Omega_2 \tag{18}$$

From Eq. (9b) one can conclude that

$$\delta\mathbf{v} = \varphi\delta\mathbf{v}_{\text{F}} \tag{19}$$

Now, combining Eqs. (17)-(19), the following relation may be written

$$\eta_0 \|\mathbf{D}\|^{2n} \mathbf{D}\ \mathbf{n} \cdot \delta\mathbf{v} + \lambda\varphi^2 \eta_0 \|\mathbf{D}_{\text{F}}\|^{2n} \mathbf{D}_{\text{F}}\ (-\mathbf{n}) \cdot \delta\mathbf{v}_{\text{F}} \ = \ 0 \tag{20}$$

and, since both $\delta\mathbf{v}$ and $\delta\mathbf{v}_{\text{F}}$ are tangent to the interface $\partial\Omega_1 \cap \partial\Omega_2$

$$\eta_0 \|\mathbf{D}\|^{2n} \mathbf{D}\ \mathbf{n} \cdot \mathbf{t} = \lambda\varphi^2 \eta_0 \|\mathbf{D}_{\text{F}}\|^{2n} \mathbf{D}_{\text{F}}\ \mathbf{n} \cdot \mathbf{t} \tag{21}$$

which is, exactly, Eq. (15). So the minimization of the functional Π was shown to be equivalent to the solution of the problem stated in Eqs. (8)-(9).

4. FINAL REMARKS

In this work the momentum transport in two distinct flow regions is modeled, in a context of a systematic local theory: the continuum theory of mixtures. In the so-called mixture region, fluid and solid (the porous matrix) are treated as continuous constituents of a binary mixture while the classical (continuum mechanics) balance equations are recovered in the pure fluid region, where the mixture is reduced to a single constituent.

Conditions at the interface pure fluid-mixture were imposed in order to assure the continuity of the velocity and shear stress fields.

Most of the works on transport phenomena in porous media use a volume-averaging technique based on a continuum mechanics approach. A different approach is used in this work. The continuum theory of mixtures, which generalizes the classical continuum mechanics, allows the construction of a thermodynamically consistent local model by means of a systematic procedure to describe the transport phenomena in two distinct flow regions: a fluid and a fluid-saturated porous medium. The cost associated with the use of the mixture theory is not high. In fact, it consists only of a few new definitions and some new terms in the balance equations such as, for instance the momentum interaction term ($\mathbf{m}_{\text{F}}$).

An equivalent minimum principle was established, providing a powerful tool for numerical simulations and for ensuring existence and uniqueness of the solution.

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