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TRANSIENT ANALYSIS OF SLIP FLOW AND HEAT TRANSFER IN MICROCHANNELS

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Abstract. *Hybrid analytical-numerical solutions for transient flow and transient convective heat transfer within microchannels are presented. Analytical solutions for flow transients in microchannels are obtained, by making use of the integral transform approach. The proposed model involves the transient fully developed flow equation for laminar regime and incompressible flow with slip at the walls, in simple channel geometries. The solution is constructed so as to account for any general functional form of the time variation of the pressure gradient along the duct. Then, transient-state convection heat transfer is solved for laminar slip flow inside microchannels formed by parallel-plates, making use of the generalized integral transform technique (GITT) and the exact analytical solution of the corresponding eigenvalue problem in terms of confluent hypergeometric functions, so as to eliminate the transversal coordinate. Then, the resulting system of transformed partial differential equations in the longitudinal coordinate is numerically solved by the Method of Lines as implemented in the routine* ***NDSolve*** *of the Mathematica system. Mixed symbolic-numerical algorithms are developed under the Mathematica platform.*

Keywords*. Forced convection, fully developed flow, transient behavior, symbolic computation, integral transforms, microchannels*

1. Introduction

The present paper addresses the transient analysis of both fluid flow and heat transfer within microchannels, in the context of fundamental work on transient forced convection aimed at the development of hybrid numerical-analytical techniques (Cotta *et al.*, 2003). The motivation is thus to extend the previously developed hybrid tools to handle both transient flow and transient convection problems in microchannels within the slip flow regime.

The analysis of internal flows in the slip-flow regime recently gained an important role in association with the fluid mechanics of various micro-electromechanical systems (MEMS) applications, as well as in the thermal control of microelectronics, as reviewed in different sources (Bayazitoglu and Tunc, 2001, Karniadakis and Beskok, 2002, Tabeling, 2003). For steady-state incompressible fully developed flow situations and laminar regime within simple geometries such as circular microtubes and parallel-plate microchannels, explicit expressions for the velocity field in terms of the Knudsen number are readily obtainable, and have been widely employed in the heat transfer analysis of microsystems, such as in (Barron *et al.*, 1997, Larrodé *et al.*, 2000, Tunc and Bayazitoglu, 2001). Only quite recently, attention has been directed to the analysis of transient flow in microchannels (Bestman *et al.*, 1995, Aubert, 1999, Yang and Kwok, 2004, Bhattacharyya *et al.*, 2003). Unsteady one-dimensional models have been extended from classical works, and analytical solutions have been sought for fully developed flows in simple geometries. These recent works are also concerned with situations in which a simple and well-defined functional form for the pressure gradient time variation is prescribed. In the classical work (Mikhailov and Ozisik, 1984), a unified solution for transient one-dimensional laminar flow models with the usual no-slip boundary condition is presented, based on the integral transform method. Their solution was then specialized to two situations: step change and periodically varying pressure gradient. The knowledge in regular size channels is therefore fairly well consolidated for models that use simple functional forms for the pressure gradient variation such as for the two cases cited above. One of the objectives of this paper is to improve and complement existing analytical solution implementations to study laminar fully developed flows in micro-ducts subjected to arbitrary source term disturbances in space and time, by making use of the Generalized Integral Transform Technique (GITT) (Cotta, 1993, Cotta and Mikhailov, 1997, Cotta, 1998, Santos *et al.*, 2001). For the purpose of achieving generality, we make use of a simple but effective analytical filtering strategy, thus yielding analytical expressions for the time and space dependence of the velocity fields in the fully developed region.

On the other hand, the heat transfer literature of the last decade has demonstrated a vivid and growing interest in thermal analysis of flows in micro-channels, both through experimental and analytical approaches, as also pointed out in recent reviews. Since the available analytical information on heat transfer in ducts could not be directly extended to flows within microchannels with wall slip, a number of contributions have been recently directed towards the analysis of internal forced convection in the micro-scale. In the paper (Barron *et al.*, 1996), the original approach in the classical work of Graetz is used to evaluate the eigenvalues for the Graetz problem extended to slip-flow. The problem considered in this paper has also an exact solution in terms of the confluent hypergeometric function, explored in Mikhailov and Cotta (1997 and 2004) to develop *Mathematica* rules for computing the desired eigenvalues with user-specified working precision. Following the work in Barron *et al.* (1996), the same technique was employed to solve the laminar flow heat convection problem in a cylindrical micro-channel with constant uniform temperature at the boundary (Larrodé *et al.*, 2000), taking into account both the velocity slip and temperature jump at the tube wall. More recently (Yu and Ameel, 2001, Turc and Bayazitoglu, 2002a), the analytical contributions were directed towards more general problem formulations, including viscous dissipation in the fluid and two-dimensional flow geometries, such as rectangular channels. For this purpose, a more flexible hybrid numerical-analytical approach was employed, based on the ideas of the Generalized Integral Transform Technique, GITT, thus avoiding more involved analysis in relation with the eigenvalue problem.

All such analysis are restricted to steady-state situations, and very little is apparently available on transient convective heat transfer within microchannels. Nevertheless, the application of unsteady phenomena in applications with MEMS devices is extremely promising (Jiang *et al.*, 2000). Then, the ability of predicting unsteady temperature fields is essential in the controlled temperature variation within the system. Only quite recently (Tunc and Bayazitoglu, 2002b), an approximate analytical solution was presented for transient convection within microchannels, for a step change on wall temperature, based on a previously proposed hybrid approach that combines the Laplace and Integral transforms concepts (Cotta and Ozisik, 1986). In this context, the second goal of this paper is thus to illustrate the results obtained from a hybrid numerical-analytical solution for temperature distributions in a fluid flowing through parallel plate microchannels, taking into account the velocity and temperature jumps at the duct surface, for the transient state. We again make use of the GITT, but invoke the exact analytical solution of the corresponding eigenvalue problem in terms of confluent hypergeometric functions (Mikhailov and Cotta, 2004), to eliminate the transversal coordinate in the original formulation. Instead of recalling the Laplace transform approach as in (Cotta and Ozisik, 1986, Tunc and Bayazitoglu, 2002b), the resulting transformed partial differential system is numerically solved by the Method of Lines, implemented within the routine **NDSolve** of the *Mathematica* system, version 4.2. As we wish to demonstrate in what follows, this combination of solution methodologies provides a very effective eigenfunction expansion behavior, through the fast converging analytical representation in the transversal coordinate, together with a flexible and fairly reliable numerical approach for the transient and longitudinal behavior of the coupled transformed potentials. The present approach complements in scope previous developments on hybrid methods for solving transient forced convection problems (Cotta and Gerk, 1994, Gondim, 1997), as recently reviewed in Gondim *et al.* (2003). The present combined algorithm makes use of both the symbolic computation capabilities and novel numerical routines introduced in recent versions of the *Mathematica* system, allowing for an updated hybrid scheme for accurately handling transient convective heat transfer under any ratio of convection and diffusion effects.

2. Analysis: Fully developed transient flow in microchannels

We consider fully developed incompressible laminar flow, accounting for slip at the walls, inside a circular microtube or a parallel plates micro-channel subjected to a pressure gradient *dp*/*dz* that varies in any arbitrary functional form with the time variable. The velocity field is represented by $u(r, t)$, which varies with the transversal coordinate, r, and time, t. The related time-dependent axial momentum equation (z-direction) is then written in dimensionless form as:

$$R^n \frac{\partial U(R,\tau)}{\partial \tau} = \frac{\partial}{\partial R}\left[R^n \frac{\partial U(R,\tau)}{\partial R}\right] + R^n P(\tau), \qquad in\ 0 < R < 1 \tag{1.a}$$

$$\left.\frac{\partial U(R,t)}{\partial R}\right|_{R=0} = 0\,; \qquad \beta^* \left.\frac{\partial U(R,t)}{\partial R}\right|_{R=1} + U(1,t) = 0\,; \qquad U(R,0) = U_0(R) \tag{1.b-d}$$

where $n = 0$ for parallel-plates, and $n = 1$ for circular tube, and we have considered the following dimensionless groups:

$$R = \frac{r}{r_1};\ \ \tau = \frac{\nu t}{r_1^2};\ \ U(R,\tau) = \frac{u(r,t)}{u_m};\ \ P(\tau) = \frac{-(dp/dz)r_1^2}{\mu u_m};\ \ \beta^* = 2\,Kn\,\beta_v\,;\ \ Kn = \lambda/2r_1\,;\ \ \beta_v = (2-\alpha_m)/\alpha_m \tag{2}$$

The generalized integral transform technique (GITT) is a well-established hybrid tool in the solution of diffusion and convection-diffusion problems, reducing to the classical integral transform analysis in classes of problems that allow for an exact treatment, like the present one. One important aspect in this kind of eigenfunction expansion approach is the convergence enhancement achievable by introducing analytical solutions that filter the original problem source terms, which are responsible for an eventual slow convergence behavior. Thus, we start the integral transformation process by obtaining the filtering solution, based on the quasi-steady version of the present problem:

$$U(R,\tau) = U_P(R,\tau) + U_h(R,\tau) \tag{3}$$

Thus, the quasi-steady formulation of the problem (1), essentially removing the transient term in eq.(1.a) is considered

$$\frac{d}{dR}\left(R^n \frac{dU_P}{dR}\right) + R^n P(\tau) = 0\,; \qquad \left.\frac{dU_P}{dR}\right|_{R=0} = 0\,; \qquad \beta^* \left.\frac{dU_P(R;t)}{dR}\right|_{R=1} + U_P(1;t) = 0 \tag{4.a-c}$$

The above ODE is directly integrated to yield the analytical filter in terms of the dimensionless time-variable pressure gradient:

$$U_P(R;\tau) = P(\tau)\frac{(1+2\beta^* - R^2)}{2(n+1)} \tag{5}$$

The resulting system for the filtered potential U_h, is then given by:

$$R^n \frac{\partial U_h}{\partial \tau} = \frac{\partial}{\partial R}(R^n \frac{\partial U_h}{\partial R}) + R^n P^*(R,\tau) \tag{6.a}$$

$$\left.\frac{\partial U_h}{\partial R}\right|_{R=0} = 0\,; \qquad \beta^* \left.\frac{\partial U_h(R,t)}{\partial R}\right|_{R=1} + U_h(1,t) = 0\,; \qquad U_h(R,0) = U_0^*(R) = U_0(R) - U_P(R,0) \tag{6.b-d}$$

where the resulting source term for the filtered system becomes

$$P^*(R,\tau) = -\frac{\partial U_P}{\partial \tau} \tag{6.e}$$

The following simple eigenvalue problem is naturally selected for the integral transformation pair construction:

$$\frac{d}{dR}(R^n \frac{d\varphi(R)}{dR}) + R^n \varphi(R) = 0\,; \qquad \left.\frac{\partial \varphi(R)}{\partial R}\right|_{R=0} = 0\,; \qquad \beta^* \left.\frac{\partial \varphi(R)}{\partial R}\right|_{R=1} + \varphi(1) = 0 \tag{7.a-c}$$

The eigenfunctions $\varphi_m(R)$ are readily obtained and given by:

$$\varphi_m(R) = \cos(\lambda_m R)\text{, for } n = 0; \qquad \text{and} \qquad \varphi_m(R) = J_0(\lambda_m R)\text{, for } n = 1 \tag{8.a, b}$$

and the related eigenvalues are computed from satisfaction of the boundary condition Eq.(7.c), while the normalization integral is analytically computed from the definition

$$N_m = \int_0^1 R^n \varphi_m^2(R) dR \tag{9}$$

The integral transform pair is written as:

$$U(R,\tau) = \sum_{m=1}^{\infty} \frac{1}{N_m} \varphi_m(R) \bar{U}_m(\tau)\text{, inverse} \tag{10.a}$$

$$\bar{U}_m(\tau) = \int_0^1 R^n \varphi_m(R)\, U(R,\tau) dR\text{, transform} \tag{10.b}$$

Operating the filtered potential Eq. (6.a) with

$$\int_0^1 \quad \varphi_m(R) dR$$

and transforming all the original potentials with the aid of the inversion formula, we obtain the following decoupled ordinary differential equations:

$$\frac{d\bar{U}_m(\tau)}{d\tau} + \lambda_m^2 \bar{U}_m(\tau) = \bar{g}_m(\tau)\text{, } \tau > 0,\ m = 1, 2, 3\ ...; \qquad \text{with} \qquad \bar{g}_m(\tau) = \int_0^1 R^n \varphi_m(R) P^*(R,\tau) dR \tag{11.a, b}$$

Similarly, the filtered initial condition (Eq. 6.d) is operated on with $\int_0^1 \quad R^n \varphi_m(R) dR$, to yield:

$$U_m(0) = \bar{f}_m\,; \qquad \bar{f}_m = \int_0^1 R^n \varphi_m(R) U_0^*(R) dR \tag{11.c, d}$$

Eqs.(11) are readily solved to yield the analytical expression for the transformed potential:

$$\bar{U}_m(\tau) = \bar{f}_m \exp(-\lambda_m^2 \tau) + \int_0^{\tau} \exp[-\lambda_m^2(\tau - \tau')] \bar{g}_m(\tau') d\tau' \tag{12}$$

Once the above solution is obtained for the transformed potential, the inversion formula, Eq. (10.a), can be used to evaluate the filtered velocity, and then the original field from Eq. (3). For computational purposes, the infinite series is evaluated to a sufficiently large finite order so as to achieve the user's requested accuracy target. The system of partial differential equations presented in Eqs. (1) was also solved in the *Mathematica 4.2* platform by making use of the built in function ***NDSolve***, with a user prescribed relative error control, for comparison with the present GITT approach.

3. Analysis: Transient convection in microchannels

Consider transient-state heat transfer in thermally developing, hydrodynamically developed forced laminar flow inside a microchannel under the following additional assumptions:

- The flow is incompressible with constant physical properties.
- Natural convection of heat is negligible.
- The entrance temperature distribution is uniform.
- The temperature at the channel wall is prescribed and uniform.

The temperature $T(y, z, t)$ of a fluid with developed velocity profile $u(y)$, flowing along the channel in the region $0 < y < r_1$, $z > 0$, is then described by the following problem in dimensionless form:

$$\frac{\partial\theta(Y,Z,\tau)}{\partial\tau}+U(Y)\frac{\partial\theta(Y,Z,\tau)}{\partial Z}=\frac{\partial^2\theta(Y,Z,\tau)}{\partial Y^2}+\frac{1}{Pe^2}\frac{\partial^2\theta(Y,Z,\tau)}{\partial Z^2}+Br\left(\frac{dU}{dY}\right)^2;\ \text{in } 0<Y<1,\ Z>0,\ \text{and } \tau>0 \quad (13.a)$$

$$\left.\frac{\partial\theta(Y,Z,\tau)}{\partial Y}\right|_{Y=0}=0;\qquad \beta\beta^*\left.\frac{\partial\theta(Y,Z,\tau)}{\partial Y}\right|_{Y=1}+\theta(1,Z,\tau)=0 \quad (13.b, c)$$

$$\theta(Y,0,\tau)=\theta_e(\tau);\qquad \left.\frac{\partial\theta(Y,Z,\tau)}{\partial Z}\right|_{Z=L}=0;\qquad \theta(Y,Z,0)=0 \quad (13.d-f)$$

where we have considered the following dimensionless groups:

$$Y=\frac{y}{r_1};\ \tau=\frac{\alpha t}{r_1^2};\ U(Y)=\frac{u(y)}{u_m};\ Pe=\frac{u_m r_1}{\alpha};\ Br=\frac{\mu u_m^2}{k\Delta T};\ \beta=\frac{\beta_t}{\beta_v};\ \theta(Y,Z,\tau)=\frac{T(z,y,t)-T_s}{\Delta T};\ Z=\frac{1}{Pe}\frac{z}{r_1} \quad (14)$$

and $\beta_t = ((2 - \alpha_t) / \alpha_t)\,(2\gamma / (\gamma + 1)) / Pr$, α_t is the thermal accommodation coefficient, λ is the molecular mean free path, $\gamma = c_p/c_v$, while c_p is specific heat at constant pressure, c_v specific heat at constant volume, and T_s is the temperature at the channel wall. The dimensionless velocity profile is given as (Mikhailov and Cotta, 2004):

$$U(Y)=\frac{6Kn\beta_v+3(1-Y^2)/2}{1+6Kn\beta_v} \quad (15)$$

The GITT solution considers a Sturm-Liouville problem that includes the velocity profile, $U(Y)$, in its formulation (Mikhailov and Cotta, 2004). This approach leads to an exact analytical solution in terms of confluent hypergeometric functions to eliminate the transversal coordinate, where $\psi_i(Y)$ are the eigenfunctions of the following Sturm-Liouville problem, with the corresponding normalization integral and normalized form of the eigenfunction:

$$\frac{d^2\psi_i(Y)}{dY^2}+\mu_i^2U(Y)\psi_i(Y)=0;\ 0<Y<1,\qquad \frac{d\psi_i(Y)}{dY}\Big|_{Y=0}=0;\qquad Kn\beta_v\beta\frac{d\psi_i(Y)}{dY}\Big|_{Y=1}=-\frac{1}{2}\psi_i(1) \quad (16.a-c)$$

$$N_i=\int_0^1 U(Y)\psi_i^2(Y)dY;\qquad \tilde{\psi}_i(Y)=\frac{\psi_i(Y)}{N_i^{1/2}} \quad (17.a, b)$$

For the proposed dimensionless velocity field in microchannels, Eq.(16.a) can be rewritten in the simpler form below, while the original eigenvalues are obtained from:

$$\frac{d^2\psi_i(Y)}{dY^2}+\nu_i^2(1+4Kn\beta_v-Y^2)\psi_i(Y)=0;\ 0<Y<1,\qquad \mu_i=\sqrt{\frac{2}{3}(1+6Kn\beta_v)}\,\nu_i,\ i=1,2,3\ldots \quad (18.a, b)$$

As discussed in Mikhailov and Cotta (2004), the solution of problem (16) is then obtained in terms of the confluent hypergeometric function, also known as Kummer function ${}_1F_1[a;b;z]$, readily available in the *Mathematica* system, as:

$$\psi_i(Y) = {}_1F_1[\frac{1-\nu_i(1+4Kn\beta_v)}{4},\frac{1}{2},\nu_i Y^2]\,e^{\nu_i\frac{Y^2}{2}} \tag{19}$$

Eq. (19) satisfies the first two Eqs. (16.a, b), and the last Eq. (16.c) thus gives the eigencondition:

$$\{2Kn\beta_v\beta\ {}_1F_1[\frac{5}{4}-\frac{\nu_i(1+4Kn\beta_v)}{4},\frac{3}{2},\nu_i]\nu_i(1-(1+4Kn\beta_v)\nu_i)+{}_1F_1[\frac{1}{4}-\frac{\nu_i(1+4Kn\beta_v)}{4},\frac{1}{2},\nu_i](1-2Kn\beta_v\beta\nu_i)\}e^{\frac{\nu_i}{2}}=0 \tag{20}$$

The left hand side of Eq. (20) defines a function of two parameters, $Kn\beta_v$ and β, which will be employed to provide the eigenvalues, ν_i, then allowing the computation of the original eigenvalues, μ_i. The next step is thus the definition of the transform-inverse pair, given by:

$$\bar{\theta}_i(Z,\tau)=\int_0^1 U(Y)\tilde{\psi}_i(Y)\,\theta(Y,Z,\tau)\,dY\text{ , transform} \tag{21.a}$$

$$\theta(Y,Z,\tau)=\sum_{i=1}^{\infty}\tilde{\psi}_i(Y)\,\bar{\theta}_i(Z,\tau)\text{, inverse} \tag{21.b}$$

Here we choose to apply the GITT on Eqs. (13) in the partial transformation strategy (Cotta and Gerk, 1994), resulting in the partial differential equations system below:

$$\sum_{j=1}^{N}A_{ij}\frac{\partial\bar{\theta}_j(Z,\tau)}{\partial\tau}+\frac{\partial\bar{\theta}_i(Z,\tau)}{\partial Z}=-\mu_i^2\,\bar{\theta}_i(Z,\tau)+\frac{1}{Pe^2}\sum_{j=1}^{N}A_{ij}\frac{\partial^2\bar{\theta}_j(Z,\tau)}{\partial Z^2}+\bar{g}_i,\quad i=1,2,...,N \tag{22.a}$$

$$\bar{\theta}_i(Z,0)=0;\qquad \bar{\theta}_i(0,\tau)=\theta_e(\tau)\bar{f}_i;\qquad \left.\frac{\partial\bar{\theta}_i(Z,\tau)}{\partial Z}\right|_{Z=L}=0 \tag{22.b-d}$$

$$A_{ij}=\int_0^1\tilde{\psi}_i(Y)\tilde{\psi}_j(Y)\,dY;\quad \bar{g}_i=\int_0^1 Br\left(\tfrac{dU}{dY}\right)^2\tilde{\psi}_i(Y)\,dY;\quad \bar{f}_i=\int_0^1 U(Y)\tilde{\psi}_i(Y)\,dY \tag{22.e-g}$$

The numerical Method of Lines as implemented in the routine **NDSolve** of the *Mathematica* system deals with system (22) by employing the default fourth order finite difference discretization in the spatial variable Z, and creating a much larger coupled system of ordinary equations for the transformed dimensionless temperature evaluated on the knots of the created mesh. This resulting system is internally solved with Gear´s method for stiff ODE systems. Once numerical results have been obtained and automatically interpolated by **NDSolve**, one can apply the inverse expression (21.b) to obtain the full dimensionless temperature field. Once $\theta(Y, Z, \tau)$ is determined from Eq. (21.b), the average temperature $\theta_{av}(Z, \tau)$ and the local Nusselt number $Nu\,(Z, \tau) = h(z, t)\,D_h/k$, where $h(z, t)$ is the heat transfer coefficient, can be found from:

$$\theta_{av}(Z,\tau)=\frac{\int_0^1 U(Y)\,\theta(Y,Z,\tau)\,dY}{\int_0^1 U(Y)\,dY};\qquad Nu(Z,\tau)=-\frac{4}{\theta_{av}(Z,\tau)}\left.\frac{\partial\theta(Y,Z,\tau)}{\partial Y}\right|_{Y=1} \tag{23.a,b}$$

4. Results and Discussion

In this section we present and discuss a few numerical results for the two problems considered, transient flow and transient convection in microchannels, which were respectively handled by the full and the partial integral transformation strategies. The aim is to demonstrate the convergence behavior within each strategy and to illustrate some physical aspects on the transient phenomena at the micro-scale. Although the developed solutions are readily applicable to different physical situations of either liquid or gas flow, we here concentrate our illustration of results on typical examples of laminar gas slip flow.

For evaluation of the constructed symbolic-numerical algorithm on transient flow analysis, we considered both geometries (parallel plates and circular tube) under two different and representative transient situations: flow start up with a step change or a periodic time variation of the pressure gradient. Here, due to space limitations, we present only a few of the parallel-plates case results (n = 0). By assigning numerical values to the parameters, β^* = 0.1, according to the chosen dimensionless formulation, we define the pressure gradient for the start-up case with a unit step change and for the periodic case, respectively as:

$$P(\tau)=\frac{3}{3\beta^*+1};\quad P(\tau)=\frac{3}{3\beta^*+1}(1-\frac{\sin(\Omega\tau)}{2}) \tag{24.a,b}$$

with $\Omega = \pi/15$ for the reported example. Table 1 below illustrates the excellent convergence characteristics of the proposed eigenfunction expansion, for the case of a periodic pressure gradient in a parallel plates channel with $\beta^* = 0.1$, and considering four different values of the dimensionless time. Truncation orders N = 10 and 30 are explicitly shown, demonstrating that six converged significant digits at least are achieved for N as low as 10. Also presented are the numerical results obtained via the method of lines implemented in the built in routine **NDSolve** of the *Mathematica* system. These results agree to within four significant digits. As was noticed along the solution procedure, the results from the integral transform solution and from the numerical built in routine are essentially coincident, since one can only observe numerical deviations in the last two significant digits. It is also observed that the analytical solution is fully converged even with less than 10 terms in the expansion. The three-dimensional plots for the velocity distribution are given below in Figure 1 for the start up and periodic cases, and we can observe the steady-state establishment, and the time variation of the dimensionless slip velocity at $y/r_1 = 1$.

Before proceeding to the analysis of transient convection with slip flow and temperature jump, we first validate the present novel strategy of combining the integral transform approach and the Method of Lines, and inspect the convergence behavior in both the partial eigenfunction expansion and the numerical procedure for the transformed partial differential system. Therefore, the test case in (Gondim, 1997, Gondim *et al.*, 2003) for a regular parallel plates channel ($Kn = 0$) is here analyzed for different and representative values of the Peclet number. It should be noted that gas flows in microchannels are likely to result in relatively low values of Reynolds number, in the range of incompressible flow modeling here adopted, which then produce Peclet numbers in a fairly wide range. Therefore, Figure 2 shows, for $Pe = 10$, the excellent agreement between the present results and the full integral transformation in refs. (Gondim, 1997, Gondim *et al.*, 2003) where a double integral transformation in both transversal and longitudinal coordinates is employed. A truncation order of just S = 15 terms was considered sufficient for convergence in the present covalidation, as we shall examine in what follows, since we are dealing with a single integral transformation, which is performed along the most diffusive direction and exactly transforms the convective term, as opposed to the double transformation in (Gondim, 1997, Gondim *et al.*, 2003) which does not yield an exact integral transformation of the convection term and requires significantly larger truncation orders.

Table 1. Convergence behavior of eigenfunction expansion for the dimensionless velocity and comparison with routine **NDSolve** (parallel plates, periodic flow, β^*=0.1).

U(R,τ); GITT with N=10, N=30, & NDSolve (Mathematica system)

Solution	***R***	***τ=5***	***τ=10***	***τ=15***	***τ=20***
GITT – N=10		0.827503	0.755419	1.31248	1.94167
GITT – N=30	0.0	0.827503	0.755419	1.31248	1.94167
NDSolve		0.82753	0.755389	1.31251	1.9416
GITT – N=10		0.716207	0.655336	1.13908	1.68375
GITT – N=30	0.4	0.716207	0.655336	1.13908	1.68375
NDSolve		0.71623	0.655309	1.13911	1.68367
GITT – N=10		0.577521	0.529949	0.921623	1.36090
GITT – N=30	0.6	0.577521	0.529949	0.921623	1.36090
NDSolve		0.577539	0.529928	0.921641	1.36084
GITT – N=10		0.136928	0.126572	0.220405	0.324602
GITT – N=30	1.0	0.136928	0.126572	0.220405	0.324602
NDSolve		0.136932	0.126567	0.220408	0.324585

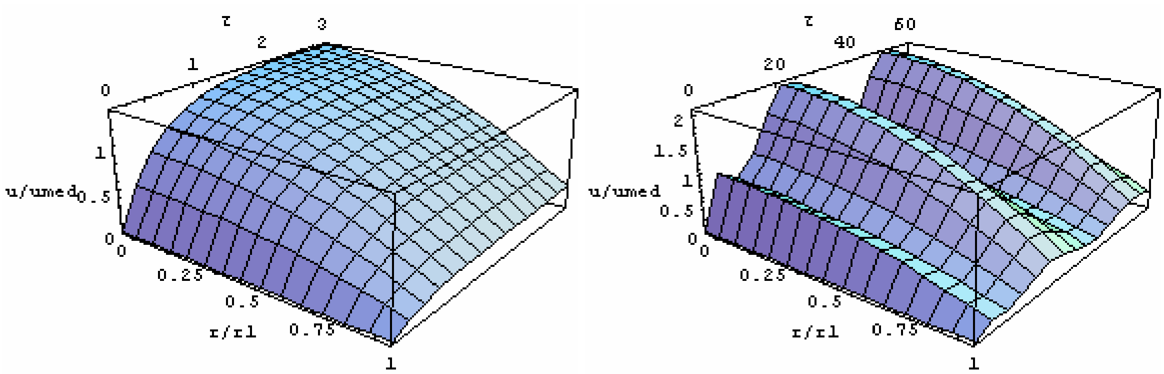

Figure 1. Transient evolution of dimensionless velocity profile for parallel-plates channel (n=0), for step change and periodic variation in pressure gradient, β^*=0.1.

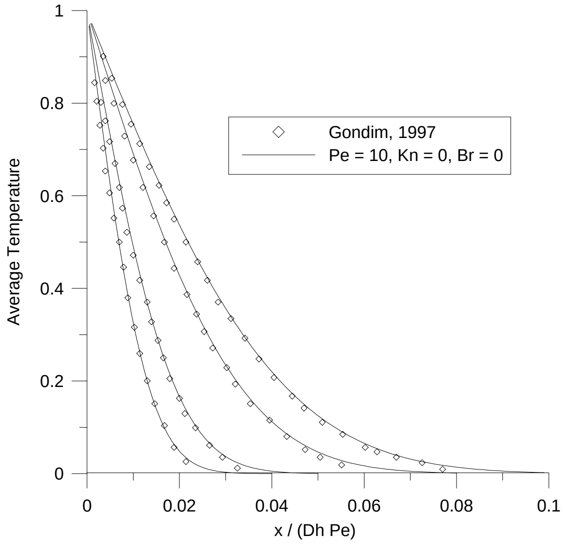

Figure 2. Transient evolution of dimensionless average temperature and covalidation with (Gondim, 1997) for parallel-plates channel and step change in inlet temperature, Pe = 10, Kn = 0, Br = 0 and S = 15. Dimensionless times ploted: τ = 0.005, 0.01, 0.03 and 0.05.

Although not likely to occur under the present physical situation of laminar gas flows in microchannels, we have considered Peclet numbers as high as 1000, in order to challenge the hybrid approach here proposed, since one expects more numerical difficulties once the convection effects predominate over the diffusion term. Thus, Tables 2 and 3 below attempt to illustrate the convergence behavior of both the eigenfunction expansion and the numerical Method of Lines in routine **NDSolve**. Table 2 for instance presents the dimensionless average temperature for different truncation orders in the eigenfunction expansion in the transversal direction, namely S = 5, 10, 15, and 20, for different longitudinal positions and time values, indicating that at least three significant digits are apparently fully converged in this range of truncation. The dimensionless average temperature distribution is practically converged to the graphical scale for truncation orders as low as S = 5. This behavior naturally offers simulations of very low computational costs and still under user controllable accuracy. Table 3, on the other hand, for a fixed value of the truncation order, S = 15, demonstrates the numerical error control built in the adopted routine, **NDSolve**, via a parameter named MaxStepSize, which controls the minimum number of nodes employed in the discretization procedure. Therefore, by decreasing the value of this parameter, we are requesting further precision to the calculation, forcing the error control to work under a more refined grid. For this example, one can observe that four significant digits are certainly unchanged by the substantial grid refinement requested.

Table 2. Convergence behavior of eigenfunction expansion for the dimensionless average temperature from partial integral transformation with routine **NDSolve** (parallel plates, Pe = 1000, Kn = 0, Br = 0, MaxStepSize = 0.0005).

	θ_m (***Pe*** **= 1000,** ***Kn*** **= 0.0)**				
	x*	**S = 5**	**S = 10**	**S = 15**	**S = 20**
t = 0.01	0.0000542	0.98468	0.98926	0.99035	0.99082
	0.0002708	0.95260	0.95141	0.95162	0.95175
	0.0004875	0.85476	0.85622	0.85632	0.85634
	0.0007042	0.64431	0.64414	0.64415	0.64415
	0.0009208	0.26693	0.26694	0.26694	0.26694
t = 0.03	0.0001667	0.97637	0.97935	0.97999	0.98028
	0.0008333	0.93310	0.93119	0.93126	0.93135
	0.0015000	0.83443	0.83682	0.83695	0.83699
	0.0021667	0.62462	0.62451	0.62452	0.62452
	0.0028333	0.14691	0.14691	0.14691	0.14691
t = 0.05	0.0002292	0.97196	0.97432	0.97485	0.97509
	0.0011458	0.92169	0.92186	0.92202	0.92213
	0.0020625	0.86591	0.86513	0.86526	0.86532
	0.0029792	0.74955	0.75042	0.75049	0.75051
	0.0038958	0.52090	0.52092	0.52092	0.52092

Table 3. Convergence behavior of Method of Lines for the dimensionless average temperature from partial integral transformation with routine **NDSolve** (parallel plates, Pe = 1000, Kn = 0, Br = 0, S = 15, MaxStepSize=MSS).

	θ_m **(*Pe* = 1000, *Kn* = 0.0)**			
	x*	**MSS= 0.001**	**MSS= 0.0005**	**MSS= 0.00025**
	0.0000542	0.98982	0.99035	0.99035
	0.0002708	0.95160	0.95162	0.95162
t = 0.01	0.0004875	0.85697	0.85632	0.85631
	0.0007042	0.64344	0.64415	0.64416
	0.0009208	0.26757	0.26694	0.26692
	0.0001667	0.97999	0.97999	0.97999
	0.0008333	0.93126	0.93126	0.93126
t = 0.03	0.0015000	0.83695	0.83695	0.83695
	0.0021667	0.62454	0.62452	0.62451
	0.0028333	0.14701	0.14691	0.14691
	0.0002292	0.97485	0.97485	0.97485
	0.0011458	0.92202	0.92202	0.92202
t = 0.05	0.0020625	0.86526	0.86526	0.86526
	0.0029792	0.75049	0.75049	0.75049
	0.0038958	0.52092	0.52092	0.52092

Transient heat transfer in microchannels is studied for typical values of the accommodation factors (α_m = 1.0 and α_t = 0.92) and just for illustration considering air as the working fluid (Pr = 0.7 and γ = 1.4). The variation of the dimensionless bulk temperature and local Nusselt number in different levels of the microscale effect is shown in Figure 3, for Pe = 10, Kn = 0, 0.001, 0.01& 0.1 and Br = 0, S =15, along the entrance region of the parallel plates channel during the transient regime. It has been observed that the bulk temperature is mildly influenced by the Knudsen number variation, according to these numerical results. However, on the Nusselt number (and thus wall heat flux), the influence is much more remarkable, with a significant increase in Nu for decreasing Kn, as the channel width increases towards the macro-scale region. This set of results also allows for the inspection of the comparative transient behavior, which indicates the less pronounced transient phenomena when the Knudsen number is increased. The microscale effects practically cease for Kn = 0.001, all along the transient behavior.

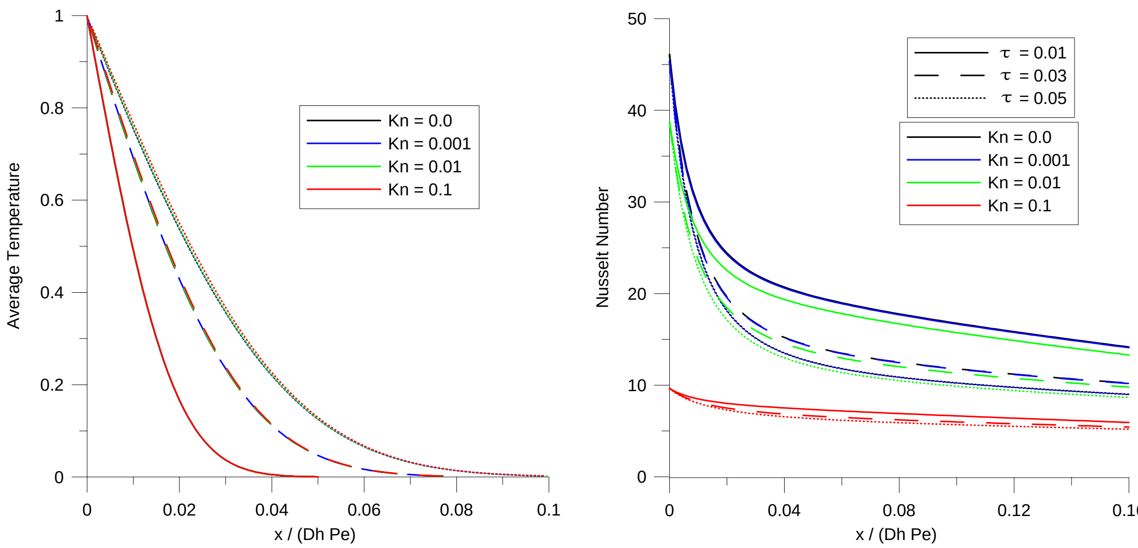

Figure 3. Influence of Knudsen number on dimensionless average temperature and local Nusselt number evolution (parallel plates, Pe = 10, Kn = 0, 0.001, 0.01& 0.1 and Br = 0, S = 15).

Figure 4 shows the effect of Brinkman number on the transient behavior of the dimensionless bulk temperature and of the local Nusselt number, for the following governing parameter values, Pe = 10, Kn = 0.01, Br = 0, 0.001, 0.005, and 0.01, S = 15. Again, the average temperature does not go through a very marked change for the different levels of frictional heating considered. The effect of increasing the Nusselt number while increasing the internal heat generation via larger values of Br, as also evident in previous steady-state analysis, is here reproduced, while the transient solutions approach such steady configurations. It can be noticed that the fluid friction heating effect is not yet noticeable at a good portion of the channel length in the earlier stages of the transient process, and it takes a certain period of time for this effect to show up at a specific longitudinal location. As time progresses, the influence of higher average temperatures deriving from the convective phenomena leads the local Nusselt at positions near the entrance to values closer to the encountered without viscous dissipation. This is also shown in Figure 4, in which we can identify the wave front position for different values of dimensionless time.

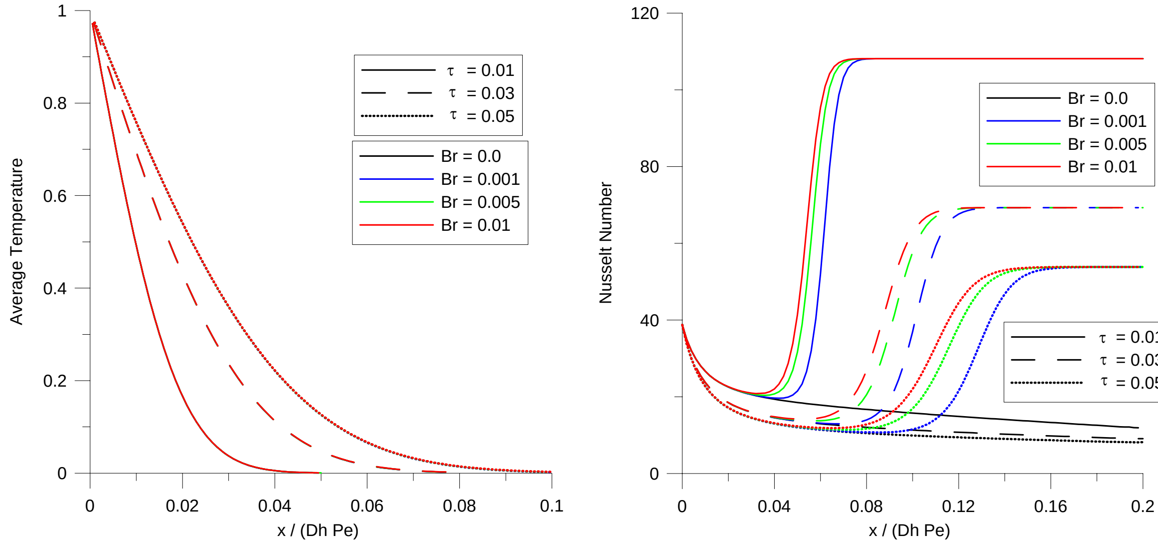

Figure 4. Influence of Brinkman number on dimensionless average temperature and local Nusselt number evolution (parallel plates, $Pe = 10$, $Kn = 0.01$, $Br = 0$, 0.001, 0.005, and 0.01, S = 15).

5. Conclusions

This work discusses hybrid numerical-analytical solutions and mixed symbolic-numerical algorithms for solving transient fully developed flow and transient forced convection in micro-channels, making use of the Generalized Integral Transform Technique (GITT) and the *Mathematica* system. The first model was described by the transient momentum equation for fully developed laminar flow of a Newtonian fluid within parallel plates and circular tubes with slip flow boundary conditions. The GITT approach proved to be very accurate and of low computational cost in solving this class of problems, due to the excellent convergence behavior provided by the time-varying filtering strategy adopted. The proposed model can be useful as a practical tool in analyzing transient flows with pressure gradient time functions fitted from experimental data, since the implementation is fully automatic for any prescribed source term input. The hybrid numerical-analytical solution for transient convection heat transfer within parallel-plates channels with laminar slip flow is also advanced, based on the integral transform approach and on the exact solution of the related eigenvalue problem, in terms of hypergeometric functions. A partial integral transformation strategy is employed, which results in a coupled system of one-dimensional partial differential equations for the transformed potentials, which are numerically handled by the Method of Lines implemented within the **NDSolve** routine of the *Mathematica* system. A symbolic-numerical implementation under the *Mathematica 4.2* platform is developed, for both the analytical and numerical computation of the related eigenfunction expansions and transformed PDE system. The approach is also readily extendable to the analysis of transient convection in micro-channels with time-varying fluid flow, in combination with the analytical solutions obtained in the first part of this work.

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