EFFECT OF VISCOUS DISSIPATION IN THERMALLY DEVELOPING FLOW IN CIRCULAR DUCTS

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Abstract. *The Classical Integral Transform Technique (CITT) is applied as solution methodology in the analysis of thermally developing flow of a non-Newtonian power-law fluid in circular tubes subjected to either prescribed wall temperature or prescribed wall heat flux. The effect of viscous dissipation is also considers to evaluate its influence in the temperature field. Results for the temperature field, as well as quantities of practical interest such as Nusselt numbers are computed for different power-law indices, which are tabulated and graphically presented as functions of the dimensionless coordinates. Critical comparisons with previous results in the literature are also performed, in order to verify the numerical codes developed in the present work and to demonstrate the consistency of the final results.*

Keywords: *Classical integral transform technique, non-Newtonian fluids, Power-law model, Viscous dissipation.*

1. INTRODUCTION

The effect of viscous dissipation is very important because is highly affects heat transfer processes whenever the fluid used has a low thermal conductivity and a high viscosity, as well as for fluid flow in small cross sectional ducts, and a small wall heat flux. Furthermore, the effect of viscous heating increases with an increase in the mass flow rate, consequently, this effect becomes more important under forced convection heat transfer. One important consequence of the effect of viscous dissipation is in the evaluation of the local Nusselt number (Dehkordi and Memari, 2010). In the context of analytical solutions in terms of eigenfunction expansions, the Classical Integral Transform Technique (CITT) appears as a reliable path for obtaining benchmark results, allowing for a more definitive critical evaluation of previously published numerical results of classical test problems.

The effect of viscous dissipation in thermally developing laminar flow have been investigated by Giudice et al. (2007) and Aydin (2005), in this latter are considered two different thermal boundary conditions: the constant heat flux and the constant wall temperature. The viscous dissipation was also considered by Jambal *et al.* (2005), Dehkordi and Memari (2010) and by Dehkordi and Mohammadi (2009) where were also investigated the effects of the power-law index on the local Nusselt number.

In this context, the present study applies the CITT in the analytical solution of the energy equation for non-Newtonian power-law fluids taking into account the effect of viscous dissipation in circular ducts which are maintained at a prescribed wall temperature or at a prescribed wall heat flux. The local Nusselt numbers are obtained with high accuracy in developing thermal region. Comparisons with previous work in the literature are also made in order to validate the numerical code developed here and to demonstrate the consistency of results produced.

2. MATHEMATICAL FORMULATION

In the analysis, we consider thermally developing flow of an incompressible non-Newtonian fluid that follows the power-law model. The circular duct is maintained at a prescribed wall temperature T_w, or at a prescribed wall heat flux q_w. The fluid enters the channels with a constant uniform temperature T_0. The energy equation in the form dimensionless including the viscous dissipation effect can be represented by:

$$v_z(r)\frac{\partial T}{\partial z} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right) + Br\left(\frac{\partial v_z}{\partial r}\right)^2 \dot{\gamma}^{n-1}, \quad 0 < r < 1, z > 0 \tag{1}$$

Subjected to the following boundary conditions:

$$T(r,0) = 1 - a\ , \ \ 0 \le r \le 1 \tag{2}$$

$$\frac{\partial T(0,z)}{\partial r} = 0, \ \ (1-a)T(1,z) + a\frac{\partial T(1,z)}{\partial r} = a\ , \ \ z > 0 \tag{3,4}$$

where,

$$v_z = v_z(r) = \frac{3n+1}{n+1}\left(1 - r^{\frac{n+1}{n}}\right); \quad \dot{\gamma} = \left|\frac{dv_z}{dr}\right| = \sqrt{\left(\frac{dv_z}{dr}\right)^2} \tag{5,6}$$

Where in the boundary conditions (3) and (4), the coefficient a identifies whether the duct wall is subjected to a prescribed temperature or to a prescribed heat flux, in the following form:

$$a = \begin{cases} 0 \ , \text{ for prescribed wall temperature;} \\ 1 \ , \text{ for prescribed wall heat flux.} \end{cases}$$

The dimensionless groups employed in the above equations are defined as:

$$r = \frac{r^*}{r_w}; \quad z = \frac{2\left(\frac{2z^*}{r_w}\right)}{\text{Re}\,\text{Pr}}; \quad v_z = \frac{v_z^*}{u_0}; \quad T = (1-a)\frac{T^* - T_w}{(T_0 - T_w)} + a\frac{T^* - T_0}{\left(\frac{q_w R_w}{K}\right)}; \quad \text{Re} = \frac{2\rho u_0^{2-n} r_w^n}{m}; \quad \text{Pr} = \frac{m}{\rho u_0^{1-n} r_w^{n-1} \alpha}$$

$$\text{Re}\,\text{Pr} = \frac{2u_o r_w}{\alpha}; \quad Br = \frac{m u_0^{n+1}}{(1-a) r_w^{n-1}(T_0 - T_w)k + a\, r_w^n q_w^{"}} \tag{7}$$

2.1. SOLUTION METHODOLOGY

To improve the computational performance is convenient define a filter that reproduces the fully developed flow solution in order to homogenize Eq. (1). Therefore, the simple filter adopted is written as:

$$T(r,z) = aT_{av}(z) + T_p(r) + T_h(r,z) \tag{8}$$

The average temperature is defined as:

$$T_{av}(z) = \frac{\int_0^1 w(r)T(r,z)dr}{\int_0^1 w(r)dr} \tag{9}$$

And for the case of a prescribed wall heat flux the average temperature is given in the form:

$$T_{av}(z) = 2z\left[1 + Br\left(\frac{3n+1}{n}\right)^n\right] \tag{10}$$

Now, introducing the Eq. (9) into Eq. (1), the following problems for the potentials $T_p(r)$ and $T_h(r,z)$ are obtained:

$$\frac{1}{r}\frac{d}{dr}\left[r\frac{dT_p(r)}{dr}\right] + Br\left(\frac{dv_z(r)}{dr}\right)^2 \dot{\gamma}^{n-1} = a\left\{v_z\left[2 + 2Br\left(\frac{3n+1}{n}\right)^n\right]\right\}, \quad 0 < r < 1 \tag{11}$$

$$\frac{dT_p(0)}{dr} = 0, \qquad (1-a)T_p(1) + a\frac{dT_p(1)}{dr} = a \tag{12,13}$$

And for prescribed wall heat flux the following additional boundary condition is considered:

$$\int_0^1 w(r)T_p(r)dr = 0 \tag{14}$$

The solution for the particular problem $T_p(r)$, for prescribed wall temperature and prescribed wall heat flux, is given, respectively, by:

$$T_p(r) = Br\left(\frac{3n+1}{n}\right)^{n-1}\left(1-r^{\frac{3n+1}{n}}\right) \tag{15}$$

$$T_p(r) = \frac{1+12n+31n^2+Br\left(\frac{3n+1}{n}\right)^n(1+8n+15n^2)}{4+32n+60n^2} - \frac{\left[1+Br\left(\frac{3n+1}{n}\right)^n\right]}{n+1}\left[\frac{(3n+1)}{2}(1-r^2)-\frac{2n^2}{3n+1}\left(1-r^{\frac{3n+1}{n}}\right)\right]+ Br\left(\frac{3n+1}{n}\right)^{n-1}\left(1-r^{\frac{3n+1}{n}}\right) \tag{16}$$

The homogeneous problem is obtained as:

$$v_z(r)\frac{\partial T_h(r,z)}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial T_r(r)}{\partial r}\right], \quad 0<r<1, z>0 \tag{17}$$

$$T_h(r,0) = (1-a) - T_p(r), \quad 0 \le r \le 1 \tag{18}$$

$$\frac{\partial T_h(0,z)}{\partial r} = 0, \ (1-a)T_h(1,z) + a\frac{\partial T_h(1,z)}{\partial r} = 0, \ z>0 \tag{19,20}$$

$$(1-a)T_h(1,z) + a\frac{\partial T_h(1,z)}{\partial r} = 0, \ z>0 \tag{20}$$

The PDE defined above by Eq. (17) can be solved by the CITT approach. Then, following the procedures of this technique, the appropriate eigenvalue problem needed for its solution are given by:

$$\frac{d}{dr}\left[r\frac{d\psi_i(r)}{dr}\right] + \mu_i^2 w(r)\psi_i(r) = 0, \quad 0<r<1 \tag{21}$$

$$\frac{d\psi_i(0)}{dr} = 0, \quad (1-a)\psi_i(1) + a\frac{d\psi_i(1)}{dr} = 0 \tag{22,23}$$

The eigenfunctions of this eigenvalue problem enjoy the following orthogonality property:

$$\int_0^1 w(r)\psi_i(r)\psi_j(r)dr = \begin{cases} 0 \ , \ i \ne j \\ N_i \ , \ i = j \end{cases} \tag{24}$$

The normalization integral N_i is then computed from:

$$N_i = \int_0^1 w(r)\psi_i^2(r)dr \tag{25}$$

where,

$$w(r) = rv_z = \left(\frac{3n+1}{n+1}\right)r\left(1-r^{\frac{n+1}{n}}\right) \tag{26}$$

To complete the solution, it is necessary to evaluate the eigenvalues, the eigenfunctions and the normalization integrals. Here, for instance, we have used the Sign-Count Method established in the works of Mikhailov and Vulchanov (1983) and Mikhailov and Özisik (1984) to determine the eigenvalues and other related eigenquantities necessary to compute the temperature field.

The eigenvalue problem defined by Eq. (21) allows for the definition of the following integral transform pair:

$$\bar{T}_i(z) = \int_0^1 w(r)\tilde{\psi}_i(r)T_h(r,z)dr, \qquad \text{Transform} \tag{27}$$

$$T_h(r,z) = \sum_{i=1}^{\infty}\tilde{\psi}_i(r)\bar{T}_i(z), \qquad \text{Inverse} \tag{28}$$

where,

$$\tilde{\psi}_i(r) = \frac{\psi_i(r)}{\sqrt{N_i}} \tag{29}$$

We can now accomplish the integral transformation of the original partial differential equation given by Eq. (1). For this purpose, Eq. (1) is multiplied by $[r\tilde{\psi}_i(r)]$ and integrated over the domain [0,1] in r and the inverse formula given by Eq. (28) is employed. After the appropriate manipulations, for the case of a prescribed wall temperature, one obtains:

$$T(r,z) = Br\left(\frac{3n+1}{n}\right)^{n-1}\left(1 - r^{\frac{3n+1}{n}}\right) + \sum_{i=1}^{\infty} \bar{h}_i \tilde{\psi}_i(r) e^{-\mu_i^2 z} \tag{30}$$

And, for prescribed wall heat flux:

$$T(r,z) = 2z\left[1 + Br\left(\frac{3n+1}{n}\right)^n\right] + \frac{1+12n+31n^2+Br\left(\frac{3n+1}{n}\right)^n\left(1+8n+15n^2\right)}{4+32n+60n^2} - \frac{\left[1+Br\left(\frac{3n+1}{n}\right)^n\right]}{n+1}\left[\frac{(3n+1)}{2}\left(1-r^2\right) - \frac{2n^2}{3n+1}\left(1-r^{\frac{3n+1}{n}}\right)\right] + Br\left(\frac{3n+1}{n}\right)^{n-1}\left(1-r^{\frac{3n+1}{n}}\right) + \sum_{i=1}^{\infty} \bar{h}_i \frac{d\tilde{\psi}_i(r)}{dr} e^{-\mu_i^2 z} \tag{31}$$

According to the definition given by Eq. (11), the average temperature for the case of prescribed wall temperature is obtained as:

$$T_{av}(z) = 2\int_0^1 w(r)T(r,z)\,dr = Br\ n^{n-1}\frac{(3n+1)^{n-1}(4n+1)}{(5n+1)} + 2\sum_{i=1}^{\infty} \bar{h}_i \bar{f}_i e^{-\mu_i^2 z} \tag{32}$$

The local Nusselt number for both situations is defined as:

$$Nu(z) = \frac{2\left.\frac{\partial T}{\partial r}\right|_{r=1}}{T(1,z) - T_{av}(z)} \tag{33}$$

Resulting the following expressions for the case of a prescribed wall temperature and for a prescribed wall heat flux, respectively:

$$Nu(z) = \frac{-2\left.\frac{\partial T}{\partial r}\right|_{r=1}}{T_{av}(z)} = \frac{2Br\left(\frac{3n+1}{n}\right)^n - 2\sum_{i=1}^{\infty} \bar{h}_i \frac{d\bar{\psi}_i(1)}{dr} e^{-\mu_i^2 z}}{Br\ n^{n-1}\frac{(3n+1)^{n-1}(4n+1)}{(5n+1)} + 2\sum_{i=1}^{\infty} \bar{h}_i \bar{f}_i e^{-\mu_i^2 z}} \tag{34}$$

$$Nu(z) = \frac{2}{T_h(1,z) + T_p(1)} = \frac{2}{\sum_{i=1}^{\infty} \bar{j}_i \bar{\psi}_i(1) + \frac{1}{4}\left[Br\left(\frac{3n+1}{n}\right)^n + \frac{1+12n+31n^2}{1+8n+15n^2}\right] e^{-\mu_i^2 z}} \tag{35}$$

where, for prescribed wall temperature:

$$\bar{h}_i = \bar{f}_i - \bar{g}_i \tag{36}$$

$$\bar{f}_i = \int_0^1 w(r)\tilde{\psi}_i(r)\,dr = -\frac{1}{\mu_i^2}\frac{d\tilde{\psi}_i(1)}{dr}\,; \quad \bar{g}_i = \int_0^1 w(r)\tilde{\psi}_i(r)T_p(r)\,dr = \frac{Br}{\mu_i^2}\left(\frac{3n+1}{n}\right)^n \int_0^1 r^{\frac{2n+1}{n}}\tilde{\psi}_i(r)\,dr \qquad (37\text{-}38)$$

and, for prescribed wall heat flux:

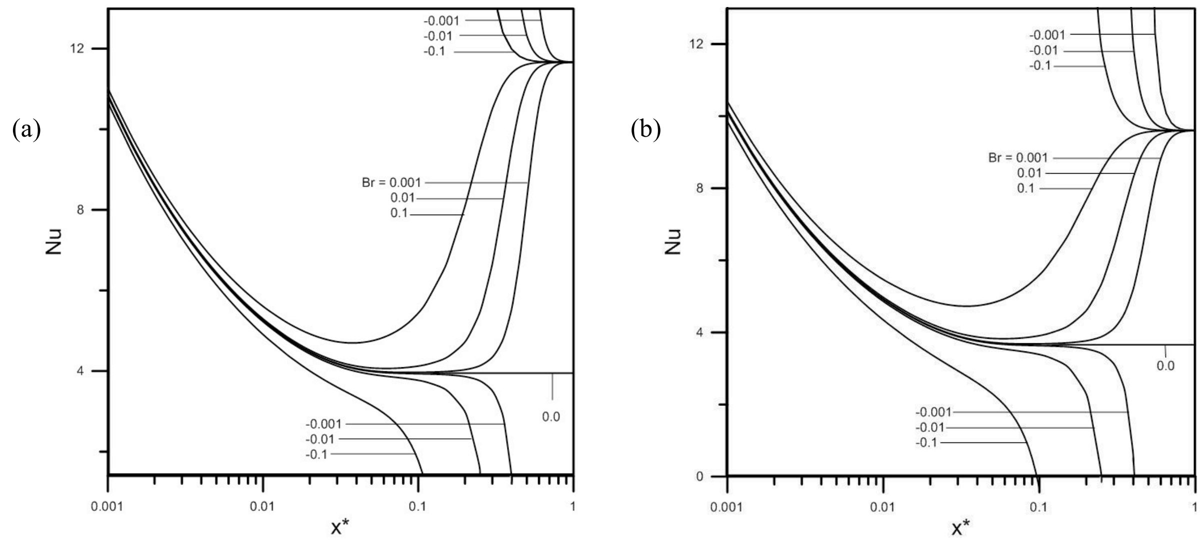
$$\bar{j}_i = -\frac{\tilde{\psi}_i(1)}{\mu_i^2} - \bar{K}_i\,; \quad \bar{K}_i = \int_0^1 w(r)\tilde{\psi}_i(r)T_p(r)\,dr = \frac{Br}{\mu_i^2}\left(\frac{3n+1}{n}\right)^{n+1} \int_0^1 r^{\frac{2n+1}{n}}\tilde{\psi}_i(r)\,dr \qquad (39,40)$$

3. RESULTS AND DISCUSSIONS

From Fig. (1), it is observed that in the absence of viscous dissipation, the solution is independent of whether there is wall heating or cooling. However, viscous dissipation always contributes to internal heating of the fluid; hence the solutions will differ according to the process taking place. As is well known, the Brinkman number accounts for the relevance of viscous dissipation. Fluid heating or cooling results in positive or negative values of Br, respectively. The thermal boundary condition considered for the tube wall was that of constant wall temperature.

Two different thermal boundary conditions have been considered for the pipe wall: constant wall temperature and constant heat flux. For each boundary condition, both wall heating and wall cooling cases are examined.

Also, the heat transfer results are illustrated in terms of conventional Nusselt numbers (Nu) against the dimensionless axial coordinate (X*) in the thermally developing region for Newtonian (n = 1), pseudoplastic (n = 0.5) and dilatant (n = 1.5) fluids with the Brinkman number as parameters.



(c)

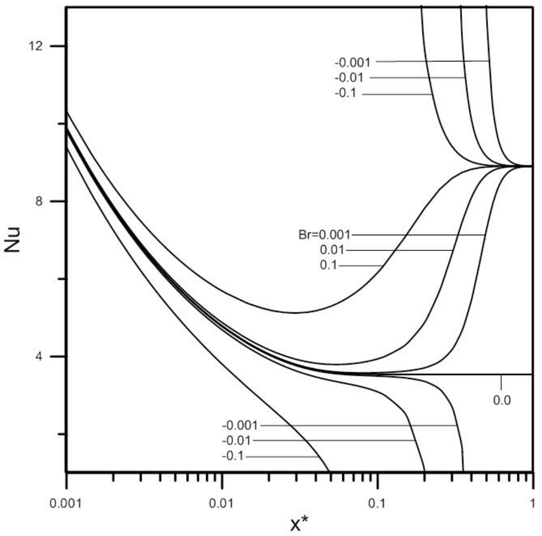


Figure 1. Effect of Brinkman number on Nusselt number for the constant wall temperature (a) n = 0.5: pseudoplastic fluid; (b) n = 1: Newtonian fluid; and (c) n = 1.5: dilatant fluid.

For the constant wall temperature case, Fig. 1, a relevant feature is that when viscous dissipation is considered ($Br \neq 0$) the asymptotic Nu is independent of Br.

As can be seen, in the case of $Br > 0$ the fluid is being cooled (inlet temperature is higher than the wall temperature). A minimum value of Nusselt number appears at some axial distance from which Nu makes a jump to its final value, because of the heat generated due to the viscous dissipation, which increases the temperature difference between the fluid and the wall and thus heat transfer. Before the jump, Nu values in the entrance region overlap with the values of $Br = 0$.

In the case of $Br < 0$ means fluid is being heated (inlet temperature is lower than the wall temperature). There exists a singular point for all curves where the Nu goes to the infinity in the thermally developing region. This is the point where the average temperature of the fluid becomes equal to the wall temperature. From this point, fluid starts to heat the wall. The fluid temperature continues to develop thermally and finally reaches the same fully developed Nu as in the case of $Br > 0$.

For the constant wall heat flux case, in Figs. 2a, 3a and 4a is possible to infer that, for small values of Brinkman number, the Nusselt number decreases monotonically from a maximum at the tube entrance to the fully developed value which is dependent upon the Brinkman number unlike the constant temperature boundary condition where the Nusselt number passed through a minimum to reach its fully developed value which is independent of the Brinkman number. In this case the heat generated by viscous dissipation and the wall heat flux are permanently present so that the fully developed Nusselt number is expected to be affected by both of them.

In Figs. 2b, 3b, 4b and 4c Nu decreases to a critical point at which the internally generated heat due to viscous dissipation balances the heat supplied by wall. After this critical point the heat generated internally suppresses the heat supplied by wall.

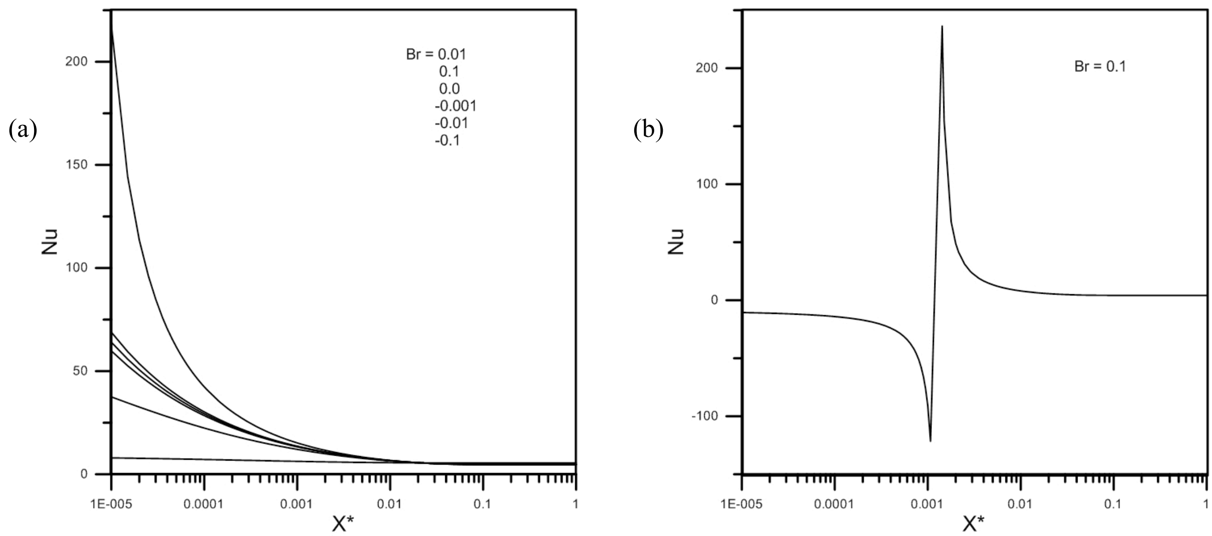


Figure 2. Effect of Brinkman number on Nusselt number for the constant wall heat flux (n = 0.5: pseudoplastic fluid).

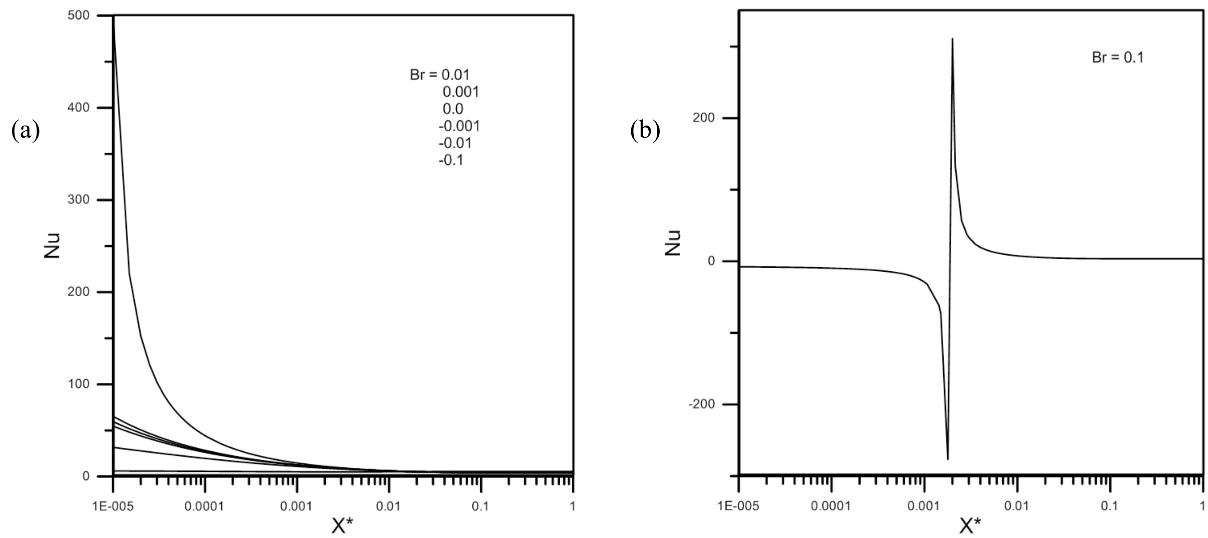
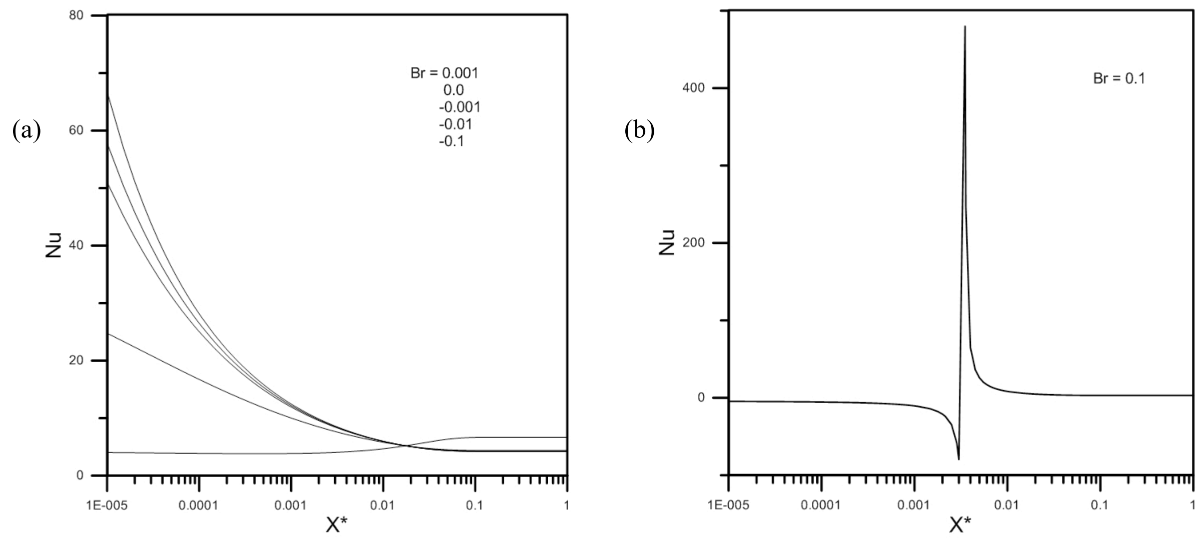


Figure 3. Effect of Brinkman number on Nusselt number for the constant wall heat flux (n = 1: Newtonian fluid).



(c)

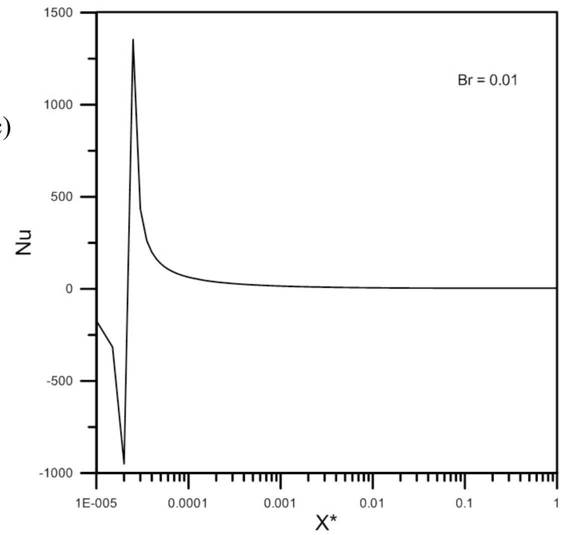


Figure 4. Effect of Brinkman number on Nusselt number for the constant wall heat flux (n = 1.5: dilatant fluid).

The numerical values of the asymptotic Nusselt numbers are given in the Tab. (1) for Br = 0 and Br ≠ 0.

Table 1. Asymptotic Nusselt numbers with different power-law indices, Brinkman numbers and dimensionless axial coordinate

Br	***Nu(z)***					
	Constant Wall Temperature			**Constant Wall Heat Flux**		
	n = 0.5	**n = 1.0**	**n = 1.5**	**n = 0.5**	**n = 1.0**	**n = 1.5**
	$X^*= 10^{-5}$					
0.1	52.6389	48.9822	47.6577	-10.5630	-7.6233	-4.6819
0.01	52,5991	48.9204	47.5552	217.8390	498.5790	-172.507
0.001	52,5951	48.9142	47.5449	68.8866	65.2572	66.7462
0.0	52.5947	48.9135	47.5438	64.0225	59.5104	57.8338
-0.001	52.5942	48.9129	47.5427	59.8000	54.6938	51.0212
-0.01	52.5902	48.9067	47.5324	37.5256	31.6437	24.7655
-0.1	52.5504	48.8449	47.4299	7.9422	6.0685	4.0295

Table 1. Continued.

Br	***Nu(z)***					
	Constant Wall Temperature			**Constant Wall Heat Flux**		
	n = 0.5	**n = 1.0**	**n = 1.5**	**n = 0.5**	**n = 1.0**	**n = 1.5**
	$X^*= 10^{-3}$					
0.1	11.0213	10.4058	10.3184	-90.9185	-29.0970	-10.4623
0.01	10.8643	10.1578	9.9049	15.2005	14.6318	15.5637
0.001	10.8485	10.1330	9.8634	13.6117	12.7202	12.4633
0.0	10.8468	10.1302	9.8588	13.4554	12.5382	12.1934
-0.001	10.8450	10.1274	9.8542	13.3027	12.3613	11.9350
-0.01	10.8293	10.1026	9.8126	12.0698	10.9687	10.0230
-0.1	10.8648	9.8532	9.3951	6.2642	5.1578	3.8520
	$X^*= 1$					
0.1	11.6666	9.5999	8.9047	4.1899	3.5821	3.0763
0.01	11.6664	9.5997	8.9046	4.6836	4.2705	4.0638
0.001	11.6644	9.5974	8.9028	7.7395	4.3541	4.1986

0.0	3.9494	3.6568	3.5392	4.7457	4.3636	4.2141
-0.001	11.6689	9.6026	8.9067	4.7521	4.3732	4.2297
-0.01	11.6669	9.6003	8.9049	4.8096	4.4609	4.3759
-0.1	11.6667	9.6003	8.9048	5.4715	5.5814	6.6874

4. CONCLUSIONS

Thermally developing flow in a circular tube has been studied taking into account the effect of viscous dissipation. The numerical results are given graphically in terms of the Nusselt number for Newtonian and power-law non-Newtonian fluids showing the effect of the Brinkman number. The wall thermal boundary conditions were considered as: constant wall temperature and constant wall heat flux. The Brinkman number is shown to play a significant role on the developing Nusselt number.

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6. RESPONSABILITY NOTICE

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