MecSol2013
International Symposium on Solid Mechanics
April 18-19, 2013
Porto Alegre, Rio Grande do Sul, Brazil

A high order theory for functionally graded shells

V.V. Zozulya
Departamento de Materials
Centro de Investigacion Cientifica de Yucatan, A.C.
e-mail: zozulya@cicy.mx

...

ABSTRACT

In this paper a new high order theory for functionally graded (FG) shells based on the expansion of the 3-D equations of elasticity for functionally graded materials (FGMs) into Fourier series in terms of Legendre's polynomials is presented. Starting from the 3-D equations of elasticity, the stress and strain tensors, the displacement, traction and body force vectors are expanded into Fourier series in terms of Legendre's polynomials in the thickness coordinate. In the same way the material parameters that describe the functionally graded material properties are also expanded into Fourier series. All equations of the linear elasticity including Hooke's law are transformed into the corresponding equations for the Fourier series expansion coefficients. Then a system of differential equations in terms of the displacements and the boundary conditions for the Fourier series expansion coefficients are obtained. In particular the first and second order approximations of the exact shell theory are considered in more details. The obtained boundary-value problems are solved by the finite element method (FEM) with COMSOL Multiphysics and MATLAB software. Numerical results are presented and discussed.

Keywords: FGMs shell, plate, rod , Legendre polynomials, FEM, power-law material graduation.

1 INTRODUCTION

Resent development of micro-electro-mechanical and nano-electro-mechanical technologies extends the field of application of the classical or non-classical theories of plates and shells towards the new thin-walled structures. The classical elasticity can be extended to the micro- and nano-scale by implementation of the theory of elasticity taking into account the physical phenomenon that can occur in such structures and devises [1, 2, 7, 11, 34].

Classical theories are based on well-known physical hypothesis; they are very popular among an engineering community because of their relative simplicity and physical clarity. Numerous books and monographs have been written in the subject, among other one can refer to [15, 19, 21, 36] But unfortunately classical theories have some shortcomings and logical contradictions such as their proximity and inaccuracy and as result in some cases not good agreement with results obtained with 3-D approach and experiments. Therefore there is demand in developing new more accurate theories.

We can mention at list two approaches to development of the theories of thin-walled structures. One consists in improvement of the classical physical hypothesis and development more accurate theories. In beams theory is well known model that take into account transversal deformations developed by Timoshenko and extended to the plate theory by Mindlin [18]. This theory was extended and applied to shells of arbitrary geometry in numerous publications and refer to as Timoshenko's theory that take into account in-plane shear deformations and rotation of the elements perpendicular to the middle surface of the shell [13, 21, 35].

The second approach consists in expansion of the stress-strain field components into polynomials series in term of thickness. It was proposed by Cauchy and Poisson and that time was not popular. Significant extension and development of that method was done by Kil'chevskii [14]. Vekua has used Legendre's polynomials for the expansion of the equations of elasticity and reduction of the 3-D problem to 2-D one [37]. Such an approach has significant advantage because of Legendre's polynomials is orthogonal and as result obtained equations are simpler. This approach was extended and applied to dynamical problems [8] and thermoelasticity [23], composite and laminate shells [24].

The approach developed in [8, 13, 23, 24, 37] has been applied to the plates and shells thermoelastic contact problems when mechanical and thermal conditions are changed during deformation in our previous publications [12, 41-50, 56- 58]. The mathematical formulation, differential equations and contact conditions for the cases of plates and cylindrical shells first time has been reported in [41, 47-49, 57, 58]. In more general form with extension to nonstationary processes and calculation all coefficients of the equations and contact and boundary conditions it was presented in [42, 43]. Then the approach was further developed to contact of plates and shells with rigid bodies though heat conducting layer [44, 45], thermoelasticity of the laminated composite materials with possibility of delamination and mechanical and thermal contact in temperature field in [44, 49, 50], the pencil-thin nuclear fuel rods modeling in [46] and some other engineering problems in [47-50, 57].

Bibliography related to different aspects of the theory and applications of the thin-walled structures contain several thousand publications, for references one can see review papers [20, 25]. For trend and recent development in the shells theory and its applications one can refer among other to books [1, 28, 30].

Functionally graded materials (FGM) are heterogeneous materials in which the elastic properties change from one surface to the other, gradually and continuously to achieve a required function [33, 35]. They attract attention especially because of their distinctive material properties, which vary continuously in one (or more) direction(s), in the case of plates and shells usually in the thickness direction [30]. FGM have been presented as an alternative to laminated composite materials that show a mismatch in properties at material interfaces. This material discontinuity in laminated composite materials leads to large inters laminar stresses and the possibility of initiation and propagation of cracks [28]. This problem is reduced in FGM because of the gradual change in mechanical properties as a function of position through the composite laminate.

The FG thin-walled structures, such as plates and shells, have numerous applications, especially in reactor vessels, turbines and many other applications in aerospace engineering [30]. Cylindrical shells have found many applications in industry [16, 32]. They are often used as load bearing structures for aircrafts, ships and buildings. The study of the stress-strain state of FG cylindrical shells is an important aspect in the successful applications of the cylindrical shells. Various theoretical models of FG plates and shells have been developed last decades [3- 6, 9, 10, 17, 26, 27, 29, 30, 38-40]. Most of the proposed models of FG plates and shells are based on the Kirchhoff-Love, Timoshenko-Mindlin [21, 22, 28] hypotheses or used more complicated high order theories such as the third-order shear deformation plate theory. Mathematically rigorous and taking into account mechanical properties important for engineering applications approach to creation high order hierarchical models of plates and shells is based on expansion of the 3-D equations of elasticity in Legendre polynomials series in term of thickness. Such an approach have been used for development various theories of isotropic [13, 37], anisotropic [23, 44], and functionally graded [51, 53, 54, 59] plates and shells.

In this paper we are developing new theory for FG shell based on expansion of the 3-D equations of elasticity for FGMs into Fourier series in terms of Legendre polynomials. More specifically, we expanded functions that describe functionally graded relations into Fourier series in terms of Legendre polynomials and find Hook's law that related Fourier coefficients for expansions of stress and strain. Then using developed in our previous publications technic we find system of differential equations and boundary conditions for Fourier coefficients. Cases of the first and second approximations have been considered in more details. Numerical examples are presented.

2 3-D FORMULATION

Let us consider a linear a linear elastic body, which occupy an open in 3-D Euclidian space simply connected bounded domain $V = \Omega \times [-h, h] \in \mathbf{R}^3$ with a smooth boundary ∂V . We assume that elastic body is FG isotropic shell of arbitrary geometry with $2h$ thickness. Boundary of the shell can be presented in the form $\partial V = S \cup \Omega^+ \cup \Omega^-$. Here Ω is the middle surface of the shell, $\partial\Omega$ is its boundary, Ω^+ and Ω^- are the outer sides and $S = \Omega \times [-h, h]$ is a sheer side.

Stress-strain state of the elastic body is defined by stress $\sigma_{ij}(\mathbf{x},t)$ and strain $\varepsilon_{ij}(\mathbf{x},t)$ tensors and displacements $u_i(\mathbf{x},t)$, traction $p_i(\mathbf{x},t)$, and body forces $b_i(\mathbf{x},t)$ vectors respectively. These quantities are not independent, they are related by equations of linear elasticity.

For convenience we introduce orthogonal system of coordinates x_1, x_2, x_3, such that position vector of arbitrary point is equal to $\mathbf{R}(x_1, x_2, x_3) = \mathbf{e}_i x_i$. Unit orthogonal basic vectors and their derivatives with respect to space coordinates are equal to

$$\mathbf{e}_i = \frac{1}{H_i}\frac{\partial \mathbf{R}}{\partial x_i}, \quad \frac{\partial \mathbf{e}_i}{\partial x_j} = \Gamma_{ij}^k \mathbf{e}_k \tag{1}$$

where H_i are Lame coefficients, Γ_{ij}^k are Christoffel symbols. They are calculated by the equations

$$H_i = \left|\frac{\partial \mathbf{R}}{\partial x_i}\right| = \sqrt{\frac{\partial \mathbf{R}}{\partial x_i}\cdot\frac{\partial \mathbf{R}}{\partial x_i}} \tag{2}$$

$$\Gamma_{ij}^k = -\frac{1}{H_i}\frac{\partial H_i}{\partial x_j}\delta_{ik} + \frac{1}{2H_iH_k}\left(\delta_{jk}\frac{\partial H_jH_k}{\partial x_i} + \delta_{ik}\frac{\partial H_iH_k}{\partial x_j} - \delta_{ij}\frac{\partial H_iH_j}{\partial x_k}\right) \tag{3}$$

From the last equation follows that $\Gamma_{ij}^k = 0$ for $i \neq j \neq k$ and

$$\Gamma_{ii}^k = -\frac{1}{H_k}\frac{\partial H_i}{\partial x_k}, \quad \Gamma_{ik}^k = -\frac{1}{H_i}\frac{\partial H_k}{\partial x_i} \quad \text{for} \quad i \neq k \tag{4}$$

In the case if displacements and their gradients are small the following kinematic Cauchy relations take place

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{1}{H_j}\frac{\partial u_i}{\partial x_j} + \frac{1}{H_i}\frac{\partial u_j}{\partial x_i}\right) + \Gamma_{ij}^k u_k \tag{5}$$

Equations of motion have the form

$$\frac{1}{H_j}\frac{\partial \sigma_{ij}}{\partial x_j} + \frac{\sigma_{ij}}{H_k}\Gamma_{ik}^k + \frac{\sigma_{ik}}{H_i}\Gamma_{ki}^j + b_i = \rho\frac{\partial^2 u_i}{\partial t^2} \tag{6}$$

Stress and strain tensors are related by generalized Hook's law, which for general anisotropic case has the form

$$\sigma^{ij}(\mathbf{x}) = c^{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}), \tag{7}$$

and for isotropic

$$c^{ijkl}(\mathbf{x}) = \lambda(\mathbf{x})g^{ij}g^{kl} + 2\mu(\mathbf{x})g^{il}g^{jk} \tag{8}$$

Here λ and μ are Lame constants, which we present in the form

$$\mu(\mathbf{x}) = \frac{E(\mathbf{x})}{2(1+\nu)},\ \lambda(\mathbf{x}) = \frac{\nu E(\mathbf{x})}{(1+\nu)(1-2\nu)} = \frac{2\mu(\mathbf{x})}{(1-2\nu)} \tag{9}$$

For convenience we transform above equations of elasticity taking into account that the radius vector $\mathbf{R}(\mathbf{x})$ of any point in domain *V,* occupied by material points of shell may be presented as

$$\mathbf{R}(\mathbf{x}) = \mathbf{r}(\mathbf{x}_\alpha) + x_3\mathbf{n}(\mathbf{x}_\alpha) \tag{10}$$

where $\mathbf{r}(\mathbf{x}_\alpha)$ is the radius vector of the points located on the middle surface of shell, $\mathbf{n}(\mathbf{x}_\alpha)$ is a unit vector normal to the middle surface.

Also we consider that $\mathbf{x}_\alpha = (x^1, x^2)$ are curvilinear coordinates associated with main curvatures of the middle surface of the shell. In this specific system of coordinates the 3-D equations of elasticity (5) - (7) can be simplified.

The equations of equilibrium have the form

$$\begin{gathered}
\frac{\partial(A_2\sigma_{11})}{\partial x_1} + \frac{\partial(A_1\sigma_{12})}{\partial x_2} + A_1A_2\frac{\partial\sigma_{13}}{\partial x_3} + \sigma_{12}\frac{\partial A_1}{\partial x_2} + \sigma_{13}A_1A_2k_1 - \sigma_{22}\frac{\partial A_2}{\partial x_1} + A_1A_2b_1 = \rho A_1A_2\frac{\partial^2 u_1}{\partial t^2}, \\
\frac{\partial(A_2\sigma_{21})}{\partial x_1} + \frac{\partial(A_1\sigma_{22})}{\partial x_2} + A_1A_2\frac{\partial\sigma_{23}}{\partial x_3} + \sigma_{21}\frac{\partial A_2}{\partial x_1} + \sigma_{23}A_1A_2k_2 - \sigma_{11}\frac{\partial A_1}{\partial x_2} + A_1A_2b_2 = \rho A_1A_2\frac{\partial^2 u_2}{\partial t^2}, \\
\frac{\partial(A_2\sigma_{31})}{\partial x_1} + \frac{\partial(A_1\sigma_{32})}{\partial x_2} + \frac{\partial(A_1A_2\sigma_{33})}{\partial x_3} - \sigma_{11}A_1A_2k_1 - \sigma_{22}A_1A_2k_2 + A_1A_2b_3 = \rho A_1A_2\frac{\partial^2 u_3}{\partial t^2}
\end{gathered} \tag{11}$$

The Cauchy relations have the form

$$\begin{gathered}
\varepsilon_{11} = \frac{1}{A_1}\frac{\partial u_1}{\partial x_1} + \frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2 + k_1u_3,\ \varepsilon_{22} = \frac{1}{A_2}\frac{\partial u_2}{\partial x_2} + \frac{1}{A_2A_1}\frac{\partial A_2}{\partial x_1}u_1 + k_2u_3, \\
\varepsilon_{33} = \frac{\partial u_3}{\partial x_3},\ \varepsilon_{12} = \frac{1}{A_2}\left(\frac{\partial u_1}{\partial x_2} - \frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2\right) + \frac{1}{A_1}\left(\frac{\partial u_2}{\partial x_1} - \frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1\right), \\
\varepsilon_{13} = \frac{\partial u_1}{\partial x_3} - k_1u_1 + \frac{1}{A_1}\frac{\partial u_3}{\partial x_1},\ \varepsilon_{23} = \frac{\partial u_2}{\partial x_3} - k_2u_2 + \frac{1}{A_2}\frac{\partial u_3}{\partial x_2}.
\end{gathered} \tag{12}$$

Here $A_\alpha(x_1, x_2) = \dfrac{\partial\mathbf{r}(x_1, x_2)}{\partial x_\alpha}$ are coefficients of the first quadratic form of a surface, $k_\alpha(x_1, x_2)$ are it main curvatures. We also take into account that shell is relatively thin and therefore

$$\frac{1}{H_\alpha}\frac{\partial H_\beta}{\partial x_\alpha} = \frac{1}{A_\alpha}\frac{\partial A_\beta}{\partial x_\alpha}, 1 + k_\alpha x_3 \approx 1 \rightarrow H_\alpha \approx A_\alpha \tag{13}$$

The generalized Hook's law have the form

$$\sigma_{ij}(\mathbf{x}) = c_{ijkl}(\mathbf{x})\varepsilon_{kl}(\mathbf{x}), \quad c_{ijkl}(\mathbf{x}) = \lambda(\mathbf{x})\delta_{ij}\delta_{kl} + \mu(\mathbf{x})(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{kj}). \tag{14}$$

Substituting the kinematic relations (12) into Hooke's law (14) and then into the equations of equilibrium (11) we obtain a system of differential equations in the terms of the displacements as

$$L_{ij}(\mathbf{x},t)u_j(\mathbf{x},t) + b_i(\mathbf{x},t) = \rho\ddot{u}_i(\mathbf{x},t), \quad \forall\mathbf{x}\in V, \quad \forall t\in\Im = [0,T] \tag{15}$$

where $L_{ij}(\mathbf{x}) = E(\mathbf{x})c^0_{ijkl}\partial_k\partial_l = E(\mathbf{x})L^0_{ij}$, L^0_{ij} is a differential operator that corresponds to the homogeneous case, upper points are partial derivatives with respect to time t.

For mathematically correct formulation of the dynamical problem of elasticity we have to formulate initial and boundary conditions. We consider the mixed boundary conditions in the form

$$\begin{aligned} u_i(\mathbf{x},t) &= \phi_i(\mathbf{x},t), \quad \forall\mathbf{x}\in\partial V_u, \quad \forall t\in\Im \\ p_i(\mathbf{x},t) &= \sigma_{ij}(\mathbf{x},t)n_j(\mathbf{x}) = P_{ij}[u_j(\mathbf{x},t)] = \psi_i(\mathbf{x},t), \quad \forall\mathbf{x}\in\partial V_p, \quad \forall t\in\Im \end{aligned} \tag{16}$$

The differential operator $P_{ij}: u_j \to p_i$ is referred to as the stress operator. It transforms the displacements into the tractions. For homogeneous anisotropic and isotropic media they have the forms

$$P_{ij} = c_{ikjl}n_k\partial_l \quad \text{and} \quad P_{ij} = \lambda\delta_{ij}\partial_n + \mu\left(n_i\partial_j + n_j\partial_i\right) \tag{17}$$

respectively. Here n_i are the components of the outward unit normal vector, $\partial_n = n_i\partial_i$ is the derivative in the direction of the vector $\mathbf{n}(\mathbf{x})$ normal to the surface ∂V_p.

Initial conditions consist of assignment of the displacements and velocity distribution in the initial moment of time. They can be written in the form

$$u_i(\mathbf{x},t_0) = u^0_i(\mathbf{x}) \quad \text{and} \quad \dot{u}_i(\mathbf{x},t_0) = v^0_i(\mathbf{x}), \quad \forall\mathbf{x}\in V \tag{18}$$

These equations will be used for the derivation of the 2-D equations for shells and plates using Fourier series in terms of Legendre's polynomials expansion.

3 2-D FORMULATION

We expand the stress-strain parameters into the Legendre polynomials series along the coordinate x_3. Such expansion can be done because of any function $f(p)$, which is defined in domain $-1\le p\le 1$ and satisfies Dirichlet's conditions (continuous, monotonous, and having finite set of discontinuity points), can be expanded into Legendre's series according formulas

$$f(p) = \sum_{k=0}^{\infty} a_k P_k(p) \quad \text{where} \quad a_n = \frac{2k+1}{2}\int_{-1}^{1} f(p)P_k(p)dp \tag{19}$$

Any function of more than one independent variable can also be expanded into Legendre's series with respect to for example, variable $x_3\in[-1,1]$, but first the new normalized variable $\omega = x_3/h\in[-1,1]$ has to be introduced. With taking into account (19) we have

$$\begin{aligned} u_i(\boldsymbol{x},t) &= \sum_{k=0}^{\infty} u^k_i(\boldsymbol{x}_\alpha,t)P_k(\omega), \quad \sigma_{ij}(\boldsymbol{x},t) = \sum_{k=0}^{\infty}\sigma^k_{ij}(\boldsymbol{x}_\alpha,t)P_k(\omega), \\ \varepsilon_{ij}(\boldsymbol{x},t) &= \sum_{k=0}^{\infty}\varepsilon^k_{ij}(\boldsymbol{x}_\alpha,t)P_k(\omega), \quad p_i(\mathbf{x},t) = \sum_{k=0}^{\infty} p^k_i(\mathbf{x}_\alpha,t)P_k(\omega) \end{aligned} \tag{20}$$

where Legendre's polynomials coefficients have the form

$$u_i^k(\boldsymbol{x}_\alpha,t)=\frac{2k+1}{2h}\int_{-h}^{h}u_i(\boldsymbol{x}_\alpha,x_3,t)P_k(\omega)dx_3,\quad \sigma_{ij}^k(\boldsymbol{x}_\alpha,t)=\frac{2k+1}{2h}\int_{-h}^{h}\sigma_{ij}(\boldsymbol{x}_\alpha,x_3,t)P_k(\omega)dx_3,$$
$$\varepsilon_{ij}^k(\boldsymbol{x}_\alpha,t)=\frac{2k+1}{2h}\int_{-h}^{h}\varepsilon_{ij}(\boldsymbol{x}_\alpha,x_3,t)P_k(\omega)dx_3,\quad p_i^k(\mathbf{x}_\alpha,t)=\frac{2k+1}{2h}\int_{-h}^{h}p_i(\mathbf{x}_\alpha,x_3,t)P_k(\omega)dx_3 \tag{21}$$

The following relations take place for the derivatives with respect to time

$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_t u_i(\boldsymbol{x}_\alpha,x_3,t)P_k(\omega)dx_3=\partial_t u_i^k(\boldsymbol{x}_\alpha,t),$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\partial_t^2 u_i(\boldsymbol{x}_\alpha,x_3,t)P_k(\omega)dx_3=\partial_t^2 u_i^k(\boldsymbol{x}_\alpha,t), \tag{22}$$

and for the derivatives with respect to coordinates $\boldsymbol{x}_\alpha$

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial u_i(\boldsymbol{x}_\alpha,x_3,t)}{\partial x_\alpha}P_k(\omega)dx_3=\frac{\partial u_i^k(\boldsymbol{x}_\alpha,t)}{\partial x_\alpha}$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \sigma_{ij}(\boldsymbol{x}_\alpha,x_3,t)}{\partial x_\alpha}P_k(\omega)dx_3=\frac{\partial \sigma_{ij}^k(\boldsymbol{x}_\alpha,t)}{\partial x_\alpha} \tag{23}$$

respectively.

Integration of the derivatives with respect to coordinates x_3 gives us

$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial u_i(\boldsymbol{x},t)}{\partial x_3}P_k(\omega)dx_3=\underline{u_i^k}(\boldsymbol{x}_\alpha,t)$$
$$\frac{2k+1}{2h}\int_{-h}^{h}\frac{\partial \sigma_{i3}(\boldsymbol{x},t)}{\partial x_3}P_k(\omega)dx_3=\frac{2k+1}{h}\left[\sigma_{i3}^+(\boldsymbol{x}_\alpha,t)-(-1)^k\sigma_{i3}^-(\boldsymbol{x}_\alpha,t)\right]-\underline{\sigma_{i3}^k}(\boldsymbol{x}_\alpha,t) \tag{24}$$

where

$$\underline{u_i^k}(\boldsymbol{x}_\alpha,t)=\frac{2k+1}{h}\left(u_i^{k+1}(\boldsymbol{x}_\alpha,t)+u_i^{k+3}(\boldsymbol{x}_\alpha,t)+\ldots\right),$$
$$\underline{\sigma_{i3}^k}(\boldsymbol{x}_\alpha,t)=A_1A_2\frac{2k+1}{h}\left(\sigma_{i3}^{k-1}(\boldsymbol{x}_\alpha,t)+\sigma_{i3}^{k-3}(\boldsymbol{x}_\alpha,t)+\ldots\right). \tag{25}$$

Now substituting stress tensor from (20) into the equations of motion (11), multiplying obtained relations by $P_k(\omega)$ and integrating over interval $[-h,h]$ with respect to x_3 we obtain 2-D equations of motion in the form

$$\frac{\partial(A_2\sigma_{11}^k)}{\partial x_1}+\frac{\partial(A_1\sigma_{12}^k)}{\partial x_2}+\sigma_{12}^k\frac{\partial A_1}{\partial x_2}+\sigma_{13}^kA_1A_2k_1-\sigma_{22}^k\frac{\partial A_2}{\partial x_1}-\underline{\sigma_{13}^k}+A_1A_2f_1^k=\rho A_1A_2\partial_t^2u_1^k,$$
$$\frac{\partial(A_2\sigma_{21}^k)}{\partial x_1}+\frac{\partial(A_1\sigma_{22}^k)}{\partial x_2}+\sigma_{12}^k\frac{\partial A_2}{\partial x_1}+\sigma_{23}^kA_1A_2k_2-\sigma_{11}^k\frac{\partial A_2}{\partial x_1}-\underline{\sigma_{23}^k}+A_1A_2f_2^k=\rho A_1A_2\partial_t^2u_2^k, \tag{26}$$
$$\frac{\partial(A_2\sigma_{31}^k)}{\partial x_1}+\frac{\partial(A_1\sigma_{32}^k)}{\partial x_2}-\sigma_{11}^kA_1A_2k_1-\sigma_{22}^kA_1A_2k_2-\underline{\sigma_{33}^k}+A_1A_2f_3^k=\rho A_1A_2\partial_t^2u_3^k.$$

where

$$f_i^k\left(\boldsymbol{x}_\alpha\right)=b_i^k\left(\boldsymbol{x}_\alpha\right)+\frac{2k+1}{h}\left(\sigma_{i3}^{+}\left(\boldsymbol{x}_\alpha\right)-(-1)^k\sigma_{i3}^{-}\left(\boldsymbol{x}_\alpha\right)\right) \tag{27}$$

In the same way can be found the 2-D kinematic Cauchy relations (12)

$$\begin{aligned}
&\varepsilon_{11}^k=\frac{1}{A_1}\frac{\partial u_1^k}{\partial x_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2^k+k_1u_3^k\,,\varepsilon_{22}^k=\frac{1}{A_2}\frac{\partial u_2^k}{\partial x_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x_1}u_1^k+k_2u_3^k\,,\\
&\varepsilon_{12}^k=\frac{1}{A_2}\left(\frac{\partial u_1^k}{\partial x_2}-\frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2^k\right)+\frac{1}{A_1}\left(\frac{\partial u_2^k}{\partial x_1}-\frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1^k\right),\\
&\varepsilon_{13}^k=\frac{1}{A_1}\frac{\partial u_3^k}{\partial x_1}-k_1u_1^k+\underline{u_1^k}\,,\varepsilon_{23}^k=\frac{1}{A_2}\frac{\partial u_3^k}{\partial x_2}-k_2u_2^k+\underline{u_2^k}\,,\varepsilon_{33}=\underline{u_3^k}.
\end{aligned} \tag{28}$$

In order to transform Hooke's law into 2-D form we expand also Young's modulus $E(\mathbf{x})$ into series of Legendre's polynomials

$$E\left(\boldsymbol{x}\right)=\sum_{r=1}^{\infty}E^r\left(x_1\right)P_r(x_3),\quad E^k\left(x_1\right)=\frac{2k+1}{2h^{1-k}}\int_{-h}^{h}E\left(x_1,x_3\right)P_k(\omega)dx_3\,. \tag{29}$$

Substituting this expansion and the expansions for the stress and strain tensors in Hooke's law (14) we obtain 2-D Hooke's law for the series expansion coefficients of Legendre's polynomials

$$\sigma_{ij}^n\left(x_1\right)=c_{ijkl}^0\sum_{r=1}^{\infty}\sum_{m}^{\infty}\in^{nrm}\,E^r\left(x_1\right)\varepsilon_{kl}^m\left(x_1\right) \tag{30}$$

where

$$\in^{nrm}=\int_{-1}^{1}P_n(x_3)P_r(x_3)P_m(x_3)dx_3 \tag{31}$$

For any specific index combination (n, r and m) the coefficients $\in^{nrm}$ can be easily calculated. Their calculations are simplified because $\in^{nrm}$ are fully symmetric with respect to their indices and many of them are equal to zero. The following formula shows how to calculate coefficients $\in^{nrm}$

$$\in^{nrm}=\begin{cases}0 & \text{for}\,n=r=0\\ 0 & \text{for}\,n=0\ \text{ and }\ r\neq m\\ (2m+1)/2 & \text{for}\ n=0\ \text{ and }\ r=m\\ 0 & \text{for}\,n=r\text{ and odd }m\text{ and }\ m>n+r\\ 0 & \text{for}\,n+r+m\text{ odd}\\ 0 & \text{for}\,n+r+m\ \text{ even and}\,m>n+r\end{cases} \tag{32}$$

In the above relations (26) - (32) the following orthogonality property of the Legendre's polynomials has been used

$$\int_{-h}^{h}P_n\left(\omega\right)P_m\left(\omega\right)dx_3=\begin{cases}\dfrac{2h}{2n+1} & \text{for}\ n=m\\ 0 & \text{for}\ n\neq m\end{cases} \tag{33}$$

In order to find 2-D differential equations in form of displacements we substitute the kinematic Cauchy relations (28) into Hooke's law for FG body (30) and substituting these equations in the equations of equilibrium (26) gives us the 2-D equations in displacements. This system of equations contains an infinite number of equations which are 2-D, they can be written in the form

$$\mathbf{E}\cdot\left(\mathbf{L}\cdot\mathbf{u}(\mathbf{x},t)\right)+\mathbf{f}(\mathbf{x},t)=\rho\ddot{\mathbf{u}}(\mathbf{x},t), \tag{34}$$

where

$$\mathbf{E}=\begin{vmatrix} E_{ij}^{00} & E_{ij}^{01} & \cdots \\ E_{ij}^{10} & E_{ij}^{11} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix},\ \mathbf{L}=\begin{vmatrix} L_{ij}^{00} & L_{ij}^{01} & \cdots \\ L_{ij}^{10} & L_{ij}^{11} & \cdots \\ \vdots & \vdots & \ddots \end{vmatrix},\ \mathbf{u}=\begin{vmatrix} u_j^0 \\ u_j^1 \\ \vdots \end{vmatrix},\ \mathbf{f}=\begin{vmatrix} f_j^0 \\ f_j^1 \\ \vdots \end{vmatrix},\ E_{ij}^{nm}=\begin{vmatrix} E^{nm} & 0 \\ 0 & E^{nm} \end{vmatrix}. \tag{35}$$

Here L_{ij}^{nm} are differential operators that correspond to homogeneous elastic shells, and $E^{nm}=\in^{nrm}E^r$ are coefficients that characterize the inhomogeneous properties of the shell.

Now instead of the finite 3-D system of the differential equations in displacements (15) we have an infinite system of 2-D differential equations for coefficients of the Legendre's polynomial series expansion. In order to simplify the problem approximate theory has to be developed and only finite number of members has to be taken into account in the expansion (20) and in all above relations. For example if we consider n-order approximate shell theory, only $n+1$ members in the expansion (20) are taken into account

$$u_i\left(\boldsymbol{x},t\right)=\sum_{k=0}^{n}u_i^k\left(\boldsymbol{x}_\alpha,t\right)P_k(\omega),\ \sigma_{ij}\left(\boldsymbol{x},t\right)=\sum_{k=0}^{n}\sigma_{ij}^k\left(\boldsymbol{x}_\alpha,t\right)P_k(\omega),\ \varepsilon_{ij}\left(\boldsymbol{x},t\right)=\sum_{k=0}^{n}\varepsilon_{ij}^k\left(\boldsymbol{x}_\alpha,t\right)P_k(\omega). \tag{36}$$

In this case we consider that $u_i^k=0$, $\sigma_{ij}^k=0$ and $\varepsilon_{ij}^k=0$ for $k<0$ and for $k>n$. Order of the system of differential equations depends on assumption regarding thickness distribution of the stress-strain parameters of the shell.

4 FIRST-ORDER THEORY

In the case if only the first two terms of the Legendre polynomials series is considered in the expansion (20) we have the first approximation the shell theory which in isotropic case usually refer as Vekua's shell theory. In this case the stress-strain parameters, which describe the state of the shell, can be presented in the form

$$\begin{aligned}
\sigma_{ij}\left(\mathbf{x},t\right)&=\sigma_{ij}^0\left(\boldsymbol{x}_\alpha,t\right)P_0\left(\omega\right)+\sigma_{ij}^1\left(\boldsymbol{x}_\alpha,t\right)P_1\left(\omega\right),\\
u_i\left(\mathbf{x},t\right)&=u_i^0\left(\boldsymbol{x}_\alpha,t\right)P_0\left(\omega\right)+u_i^1\left(\mathbf{x}_\nu,t\right)P_1\left(\omega\right),\\
\varepsilon_{ij}\left(\mathbf{x},t\right)&=\varepsilon_{ij}^0\left(\boldsymbol{x}_\alpha,t\right)P_0\left(\omega\right)+\varepsilon_{ij}^1\left(\boldsymbol{x}_\alpha,t\right)P_1\left(\omega\right),\\
p_i\left(\mathbf{x},t\right)&=p_i^0\left(\boldsymbol{x}_\alpha,t\right)P_0\left(\omega\right)+p_i^1\left(\boldsymbol{x}_\alpha,t\right)P_1\left(\omega\right).
\end{aligned} \tag{37}$$

where coefficients of the expansion are

$$
\begin{aligned}
&u_i^0\left(\boldsymbol{x}_\alpha,t\right)=\frac{1}{2h}\int_{-h}^{h}u_i\left(\boldsymbol{x}_\alpha,x_3,t\right)dx_3, \quad u_i^1\left(\boldsymbol{x}_\alpha,t\right)=\frac{3}{2h}\int_{-h}^{h}u_i\left(\boldsymbol{x}_\alpha,x_3,t\right)x_3dx_3\\
&\varepsilon_{ij}^0\left(\boldsymbol{x}_\alpha,t\right)=\frac{1}{2h}\int_{-h}^{h}\varepsilon_{ij}\left(\boldsymbol{x}_\alpha,x_3,t\right)dx_3, \quad \varepsilon_{ij}^1\left(\boldsymbol{x}_\alpha,t\right)=\frac{3}{2h}\int_{-h}^{h}\varepsilon_{ij}\left(\boldsymbol{x}_\alpha,x_3,t\right)x_3dx_3\\
&\sigma_{ij}^0\left(\boldsymbol{x}_\alpha,t\right)=\frac{1}{2h}\int_{-h}^{h}\sigma_{ij}\left(\boldsymbol{x}_\alpha,x_3,t\right)dx_3, \quad \sigma_{ij}^1\left(\boldsymbol{x}_\alpha,t\right)=\frac{3}{2h}\int_{-h}^{h}\sigma_{ij}\left(\boldsymbol{x}_\alpha,x_3,t\right)x_3dx_3\\
&p_i^0\left(\boldsymbol{x}_\alpha,t\right)=\frac{1}{2h}\int_{-h}^{h}p_i\left(\boldsymbol{x}_\alpha,x_3,t\right)dx_3, \quad p_i^1\left(\boldsymbol{x}_\alpha,t\right)=\frac{3}{2h}\int_{-h}^{h}p_i\left(\boldsymbol{x}_\alpha,x_3,t\right)x_3dx_3
\end{aligned}
\tag{38}
$$

The equations of motion (26) in this case have the form

$$
\begin{aligned}
&\frac{\partial\left(A_2\sigma_{11}^0\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{12}^0\right)}{\partial x_2}+\sigma_{12}^0\frac{\partial A_1}{\partial x_2}+\sigma_{13}^0A_1A_2k_1-\sigma_{22}^0\frac{\partial A_2}{\partial x_1}-\underline{\sigma_{13}^0}+A_1A_2f_1^0=\rho A_1A_2\ddot{u}_1^0\ ,\\
&\frac{\partial\left(A_2\sigma_{21}^0\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{22}^0\right)}{\partial x_2}+\sigma_{12}^0\frac{\partial A_2}{\partial x_1}+\sigma_{23}^0A_1A_2k_2-\sigma_{11}^0\frac{\partial A_2}{\partial x_1}-\underline{\sigma_{23}^0}+A_1A_2f_2^0=\rho A_1A_2\ddot{u}_2^0\ ,\\
&\frac{\partial\left(A_2\sigma_{31}^0\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{32}^0\right)}{\partial x_2}-\sigma_{11}^0A_1A_2k_1-\sigma_{22}^0A_1A_2k_2-\underline{\sigma_{33}^0}+A_1A_2f_3^0=\rho A_1A_2\ddot{u}_3^0\ .\\
&\frac{\partial\left(A_2\sigma_{11}^1\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{12}^1\right)}{\partial x_2}+\sigma_{12}^1\frac{\partial A_1}{\partial x_2}+\sigma_{13}^1A_1A_2k_1-\sigma_{22}^1\frac{\partial A_2}{\partial x_1}-\underline{\sigma_{13}^1}+A_1A_2f_1^1=\rho A_1A_2\ddot{u}_1^1\ ,\\
&\frac{\partial\left(A_2\sigma_{21}^1\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{22}^1\right)}{\partial x_2}+\sigma_{12}^1\frac{\partial A_2}{\partial x_1}+\sigma_{23}^1A_1A_2k_2-\sigma_{11}^1\frac{\partial A_2}{\partial x_1}-\underline{\sigma_{23}^1}+A_1A_2f_2^1=\rho A_1A_2\ddot{u}_2^1\ ,\\
&\frac{\partial\left(A_2\sigma_{31}^1\right)}{\partial x_1}+\frac{\partial\left(A_1\sigma_{32}^1\right)}{\partial x_2}-\sigma_{11}^1A_1A_2k_1-\sigma_{22}^1A_1A_2k_2-\underline{\sigma_{33}^1}+A_1A_2f_3^1=\rho A_1A_2\ddot{u}_3^1\ .
\end{aligned}
\tag{39}
$$

The kinematic Cauchy relations (28) have the form

$$
\begin{aligned}
&\varepsilon_{11}^0=\frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2^0+k_1u_3^0\ ,\varepsilon_{22}^0=\frac{1}{A_2}\frac{\partial u_2^0}{\partial x_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x_1}u_1^0+k_2u_3^0\ ,\\
&\varepsilon_{12}^0=\frac{1}{A_2}\left(\frac{\partial u_1^0}{\partial x_2}-\frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2^0\right)+\frac{1}{A_1}\left(\frac{\partial u_2^0}{\partial x_1}-\frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1^0\right),\\
&\varepsilon_{13}^0=\frac{1}{A_1}\frac{\partial u_3^0}{\partial x_1}-k_1u_1^0+\frac{1}{h}u_1^1,\varepsilon_{23}^0=\frac{1}{A_2}\frac{\partial u_3^0}{\partial x_2}-k_2u_2^0+\frac{1}{h}u_2^1,\varepsilon_{33}^0=\frac{1}{h}u_3^1.\\
&\varepsilon_{11}^1=\frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1}+\frac{1}{A_1A_2}\frac{\partial A_1}{\partial x_2}u_2^1+k_1u_3^1\ ,\varepsilon_{22}^1=\frac{1}{A_2}\frac{\partial u_2^1}{\partial x_2}+\frac{1}{A_1A_2}\frac{\partial A_2}{\partial x_1}u_1^1+k_2u_3^1\ ,\\
&\varepsilon_{12}^1=\frac{1}{A_2}\left(\frac{\partial u_1^1}{\partial x_2}-\frac{1}{A_1}\frac{\partial A_2}{\partial x_1}u_2^1\right)+\frac{1}{A_1}\left(\frac{\partial u_2^1}{\partial x_1}-\frac{1}{A_2}\frac{\partial A_1}{\partial x_2}u_1^1\right),\\
&\varepsilon_{13}^1=\frac{1}{A_1}\frac{\partial u_3^1}{\partial x_1}-k_1u_1^1\ ,\varepsilon_{23}^1=\frac{1}{A_2}\frac{\partial u_3^1}{\partial x_2}-k_2u_2^1\ ,\varepsilon_{33}^1=0.
\end{aligned}
\tag{40}
$$

The generalized Hooke's law for FG material (30) in this case has the form

$$\begin{aligned}\sigma_{ij}^0 &= c_{ijkl}^0\left(\in^{000}\ E^0\varepsilon_{kl}^0+\in^{001}\ E^0\varepsilon_{kl}^1+\in^{010}\ E^1\varepsilon_{kl}^0+\in^{011}\ E^1\varepsilon_{kl}^1\right)\\ \sigma_{ij}^1 &= c_{ijkl}^0\left(\in^{100}\ E^0\varepsilon_{kl}^0+\in^{101}\ E^0\varepsilon_{kl}^1+\in^{110}\ E^1\varepsilon_{kl}^0+\in^{111}\ E^1\varepsilon_{kl}^1\right)\end{aligned} \tag{41}$$

where

$$\in^{000}=2\,,\in^{001}=0\,,\in^{010}=0\,,\in^{011}=\frac{2}{3}\,,\in^{100}=0\,,\in^{101}=\frac{2}{3}\,,\in^{110}=\frac{2}{3}\,,\in^{111}=0\,. \tag{42}$$

Hence the generalized Hooke's law (41) takes the form

$$\sigma_{ij}^0 = c_{ijkl}^0\left(2E^0\varepsilon_{kl}^0+\frac{2}{3}E^1\varepsilon_{kl}^1\right),\ \sigma_{ij}^1 = c_{ijkl}^0\left(\frac{2}{3}E^1\varepsilon_{kl}^0+\frac{2}{3}E^0\varepsilon_{kl}^1\right) \tag{43}$$

Substitution of the kinematic Cauchy relations (40) into the generalized Hooke's law (43) and the result to the equations of motion (39) gives us the 2-D equations in displacements for the first-order FG shell theory in the form (34), but now it contains only six equations and the corresponding matrixes and vectors have the form

$$\mathbf{E}=\begin{vmatrix}E_{ij}^{00} & E_{ij}^{01}\\ E_{ij}^{10} & E_{ij}^{11}\end{vmatrix},\mathbf{L}=\begin{vmatrix}L_{ij}^{00} & L_{ij}^{01}\\ L_{ij}^{10} & L_{ij}^{11}\end{vmatrix},\mathbf{u}=\begin{vmatrix}u_j^0\\ u_j^1\end{vmatrix},\mathbf{f}=\begin{vmatrix}f_j^0\\ f_j^1\end{vmatrix}. \tag{44}$$

Differential operators that appear in the equations (44) for shells of arbitrary geometry are presented in [].

5 SECOND-ORDER THEORY

In the second order approximation only the first three series terms of the Legendre's polynomials have to be taken into account. In this case the stress and strain state of the shell, can be represented in the form

$$\begin{aligned}\sigma_{ij}(\mathbf{x},t)&=\sigma_{ij}^0(\boldsymbol{x}_\alpha,t)P_0(\omega)+\sigma_{ij}^1(\boldsymbol{x}_\alpha,t)P_1(\omega)+\sigma_{ij}^2(\boldsymbol{x}_\alpha,t)P_2(\omega),\\ u_i(\mathbf{x},t)&=u_i^0(\boldsymbol{x}_\alpha,t)P_0(\omega)\ +u_i^1(\mathbf{x}_\nu,t)P_1(\omega)+u_i^2(\mathbf{x}_\nu,t)P_2(\omega),\\ \varepsilon_{ij}(\mathbf{x},t)&=\varepsilon_{ij}^0(\boldsymbol{x}_\alpha,t)P_0(\omega)+\varepsilon_{ij}^1(\boldsymbol{x}_\alpha,t)P_1(\omega)+\varepsilon_{ij}^2(\boldsymbol{x}_\alpha,t)P_2(\omega),\\ p_i(\mathbf{x},t)&=p_i^0(\boldsymbol{x}_\alpha,t)P_0(\omega)\ +p_i^1(\boldsymbol{x}_\alpha,t)P_1(\omega)+p_i^2(\boldsymbol{x}_\alpha,t)P_2(\omega).\end{aligned} \tag{45}$$

The Legendre's polynomials coefficients in (45) can be calculated using (21). The equations of motions and Cauchy relations can be calculated from (26) and (28) respectively.

The generalized Hooke's law for FG material (30) with taking into account

$$\begin{aligned}&\in^{000}=2\,,\in^{001}=0\,,\in^{002}=0\,,\in^{010}=0\,,\in^{011}=\frac{2}{3}\,,\in^{012}=0\,,\in^{020}=0\,,\in^{021}=0\,,\in^{022}=\frac{2}{5}\,,\\ &\in^{100}=0\,,\in^{101}=\frac{2}{3}\,,\in^{102}=0\,,\in^{110}=\frac{2}{3}\,,\in^{111}=0\,,\in^{112}=\frac{4}{15}\,,\in^{120}=0\,,\in^{121}=\frac{4}{15}\,,\in^{122}=0\,,\\ &\in^{200}=0\,,\in^{201}=0\,,\in^{202}=\frac{2}{5}\,,\in^{210}=0\,,\in^{211}=\frac{4}{15}\,,\in^{212}=0\,,\in^{220}=\frac{2}{5}\,,\in^{221}=0\,,\in^{222}=\frac{4}{35}\,.\end{aligned} \tag{46}$$

in this case has the form

$$\sigma_{ij}^{0}=c_{ijkl}^{0}\left(2E^{0}\varepsilon_{kl}^{0}+\frac{2}{3}E^{1}\varepsilon_{kl}^{1}+\frac{2}{5}E^{2}\varepsilon_{kl}^{2}\right),$$
$$\sigma_{ij}^{1}=c_{ijkl}^{0}\left(\frac{2}{3}E^{1}\varepsilon_{kl}^{0}+\frac{2}{3}E^{0}\varepsilon_{kl}^{1}+\frac{4}{15}E^{2}\varepsilon_{kl}^{1}+\frac{4}{15}E^{1}\varepsilon_{kl}^{2}\right), \tag{47}$$
$$\sigma_{ij}^{2}=c_{ijkl}^{0}\left(\frac{2}{5}E^{2}\varepsilon_{kl}^{0}+\frac{4}{15}E^{1}\varepsilon_{kl}^{1}+\frac{2}{5}E^{0}\varepsilon_{kl}^{2}+\frac{4}{35}E^{2}\varepsilon_{kl}^{2}\right).$$

In the second-order theory the system of equations for the displacements has the same form as (34), but it contains only nine equations and the corresponding matrixes and vector can be written as

$$\mathbf{E}=\begin{vmatrix}E_{ij}^{00} & E_{ij}^{01} & E_{ij}^{02}\\ E_{ij}^{10} & E_{ij}^{11} & E_{ij}^{12}\\ E_{ij}^{20} & E_{ij}^{21} & E_{ij}^{22}\end{vmatrix},\mathbf{L}=\begin{vmatrix}L_{ij}^{00} & L_{ij}^{01} & L_{ij}^{02}\\ L_{ij}^{10} & L_{ij}^{11} & L_{ij}^{12}\\ L_{ij}^{20} & L_{ij}^{21} & L_{ij}^{22}\end{vmatrix},\mathbf{u}=\begin{vmatrix}u_j^0\\ u_j^1\\ u_j^2\end{vmatrix},\mathbf{f}=\begin{vmatrix}f_j^0\\ f_j^1\\ f_j^2\end{vmatrix}. \tag{48}$$

Differential operators that appear in the equations (48) for shells of arbitrary geometry are presented in [36].

6 AXISYMMETRIC CYLINDRICAL SHELL

All equations for the case of axisymmetric cylindrical shell can be obtained from general equations presented here. In this case components of the stress-strain state do not depend of coordinate x_2 and have the form

$$\sigma_{ij}=\begin{vmatrix}\sigma_{11} & 0 & \sigma_{13}\\ 0 & \sigma_{22} & 0\\ \sigma_{31} & 0 & \sigma_{33}\end{vmatrix},\ \varepsilon_{ij}=\begin{vmatrix}\varepsilon_{11} & 0 & \varepsilon_{13}\\ 0 & \varepsilon_{22} & 0\\ \varepsilon_{31} & 0 & \varepsilon_{33}\end{vmatrix},\ u_i=\begin{vmatrix}u_1\\ 0\\ u_3\end{vmatrix} \tag{49}$$

and can be represented in the form

$$u_i(\boldsymbol{x})=\sum_{k=0}^{\infty}u_i^k(x_1)P_k(\omega),\ u_i^k(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}u_i(x_1,x_3)P_k(\omega)dx_3,$$
$$\sigma_{ij}(\boldsymbol{x})=\sum_{k=0}^{\infty}\sigma_{ij}^k(x_1)P_k(\omega),\ \sigma_{ij}^k(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}\sigma_{ij}(x_1,x_3)P_k(\omega)dx_3,$$
$$\varepsilon_{ij}(\boldsymbol{x})=\sum_{k=0}^{\infty}\varepsilon_{ij}^k(x_1)P_k(\omega),\ \varepsilon_{ij}^k(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}\varepsilon_{ij}(x_1,x_3)P_k(\omega)dx_3, \tag{50}$$
$$p_i(\boldsymbol{x})=\sum_{k=0}^{\infty}p_i^k(x_1)P_k(\omega),\ p_i^k(x_1)=\frac{2k+1}{2h}\int_{-h}^{h}p_i(x_1,x_3)P_k(\omega)dx_3.$$

In order to obtain high order 1-D differential equations for axisymmetric cylindrical shell we have to assign in all above equations $A_1=1, A_2=R, k_1=0, k_2=k=1/R$, where R is a radius of the shell.

6.1 First-order theory

In the first order approximation theory only the first two series terms of the Legendre's polynomials have to be taken into account. In the homogeneous case it is usually referred to as the

Vekua's theory of shells. In this case the stress and strain state of the shell, can be expressed in the form

$$\begin{aligned}
\sigma_{ij}(\boldsymbol{x}) &= \sigma_{ij}^0(x_1)P_0(\omega)+\sigma_{ij}^1(x_1)P_1(\omega),\\
\varepsilon_{ij}(\boldsymbol{x}) &= \varepsilon_{ij}^0(x_1)P_0(\omega)+\varepsilon_{ij}^1(x_1)P_1(\omega),\\
p_i(\boldsymbol{x}) &= p_i^0(x_1)P_0(\omega)+p_i^1(x_1)P_1(\omega),\\
b_i(\boldsymbol{x}) &= b_i^0(x_1)P_0(\omega)+b_i^1(x_1)P_1(\omega).
\end{aligned} \tag{51}$$

The system of equations for the displacements has the same form as (34), but it contains only four equations and the corresponding matrixes and vectors have the form

$$\mathbf{E}=\begin{vmatrix} 2E^0 & 0 & \frac{2}{3}E^1 & 0\\ 0 & 2E^0 & 0 & \frac{2}{3}E^1\\ \frac{2}{3}E^1 & 0 & \frac{2}{3}E^0 & 0\\ 0 & \frac{2}{3}E^1 & 0 & \frac{2}{3}E^0 \end{vmatrix},\mathbf{L}=\begin{vmatrix} L_{11}^{00} & L_{13}^{00} & 0 & L_{13}^{01}\\ L_{31}^{00} & L_{33}^{00} & L_{31}^{01} & L_{33}^{01}\\ 0 & L_{13}^{10} & L_{11}^{11} & L_{13}^{11}\\ L_{31}^{10} & L_{33}^{10} & L_{31}^{11} & L_{33}^{11} \end{vmatrix},\mathbf{u}=\begin{vmatrix} u_1^0\\ u_3^0\\ u_1^1\\ u_3^1 \end{vmatrix},\mathbf{f}=\begin{vmatrix} f_1^0\\ f_3^0\\ f_1^1\\ f_3^1 \end{vmatrix}. \tag{52}$$

Most of the operators L_{ij}^{nm} are differential operators and they are given by

$$\begin{aligned}
&L_{11}^{00}u_1^0=(\lambda+2\mu)\frac{\partial^2 u_1^0}{\partial x_1^2},\, L_{13}^{00}u_3^0=\frac{\lambda}{R}\frac{\partial u_3^0}{\partial x_1}, L_{11}^{01}u_1^1=0, L_{13}^{01}u_3^1=\frac{\lambda}{h}\frac{\partial u_3^1}{\partial x_1},\\
&L_{31}^{00}u_1^0=-\frac{\lambda}{R}\frac{\partial u_1^0}{\partial x_1}, L_{33}^{00}u_3^0=\mu\frac{\partial^2 u_3^0}{\partial x_1^2}-(\lambda+2\mu)\frac{u_3^0}{R}, L_{31}^{01}=\frac{\mu}{h}\frac{\partial u_1^1}{\partial x_1}, L_{33}^{01}=-\frac{\lambda}{Rh}u_3^1,\\
&L_{11}^{10}u_1^0=0, L_{13}^{10}u_3^0=-\frac{3\mu}{Rh}\frac{\partial u_3^0}{\partial x_1}, L_{11}^{11}u_1^1=(\lambda+2\mu)\frac{\partial^2 u_1^1}{\partial x_1^2}-\frac{3\mu}{h^2}u_1^1, L_{13}^{11}u_3^1=\frac{\lambda}{R}\frac{\partial u_3^1}{\partial x_1}, L_{13}^{11}u_3^1=\frac{\lambda}{R}\frac{\partial u_3^1}{\partial x_1},\\
&L_{31}^{10}u_1^0=-\frac{3\lambda}{h}\frac{\partial u_1^0}{\partial x_1}, L_{33}^{10}u_3^0=-\frac{3\lambda}{h}u_3^0, L_{31}^{11}u_1^1=-\frac{\lambda}{R}\frac{\partial u_1^1}{\partial x_1}, L_{33}^{11}u_3^1=\mu\frac{\partial^2 u_3^1}{\partial x_1^2}-(\lambda+2\mu)\left(\frac{1}{R^2}-\frac{3R}{h^2}\right)u_3^1.
\end{aligned} \tag{53}$$

Components of strain and stress tensors can be calculated using relations

$$\begin{aligned}
&\varepsilon_{11}^0=\frac{\partial u_1^0}{\partial x_1}, \varepsilon_{22}^k=\frac{u_3^0}{R}, \varepsilon_{33}^0=\frac{1}{h}u_3^1, \varepsilon_{13}^0=\frac{\partial u_3^0}{\partial x_1}+\frac{1}{h}u_1^1,\\
&\varepsilon_{11}^1=\frac{\partial u_1^1}{\partial x_1}, \varepsilon_{22}^1=\frac{u_3^1}{R}, \varepsilon_{33}^1=0, \varepsilon_{13}^1=\frac{\partial u_3^1}{\partial x_1},
\end{aligned} \tag{54}$$

$$\sigma_{ij}^0=c_{ijkl}^0\left(2E^0\varepsilon_{kl}^0+\frac{2}{3}E^1\varepsilon_{kl}^1\right), \sigma_{ij}^1=c_{ijkl}^0\left(\frac{2}{3}E^1\varepsilon_{kl}^0+\frac{2}{3}E^0\varepsilon_{kl}^1\right), \tag{55}$$

and (51). Here

$$c_{ijkl}^0=\lambda^0\delta_{ij}\delta_{kl}+\mu^0\left(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk}\right) \tag{56}$$

6.2 Second-order theory

In the second order approximation only the first three series terms of the Legendre's polynomials have to be taken into account. In this case the stress and strain state of the shell, can be represented in the form

$$\begin{aligned}
\sigma_{ij}(\boldsymbol{x}) &= \sigma_{ij}^{0}(x_1)P_0(\omega)+\sigma_{ij}^{1}(x_1)P_1(\omega)+\sigma_{ij}^{2}(x_1)P_2(\omega),\\
\varepsilon_{ij}(\boldsymbol{x}) &= \varepsilon_{ij}^{0}(x_1)P_0(\omega)+\varepsilon_{ij}^{1}(x_1)P_1(\omega)+\varepsilon_{ij}^{2}(x_1)P_2(\omega),\\
u_i(\boldsymbol{x}) &= u_i^{0}(x_1)P_0(\omega)+u_i^{1}(x_1)P_1(\omega)+u_i^{2}(x_1)P_2(\omega),\\
p_i(\boldsymbol{x}) &= p_i^{0}(x_1)P_0(\omega)+p_i^{1}(x_1)P_1(\omega)+p_i^{2}(x_1)P_2(\omega).
\end{aligned} \tag{57}$$

The system of equations for the displacements has the same form as (34), but it contains only four equations and the corresponding matrixes and vectors have the form

$$\mathbf{E}=\begin{vmatrix}
2E^0 & 0 & \frac{2}{3}E^1 & 0 & \frac{2}{5}E^2 & 0\\
0 & 2E^0 & 0 & \frac{2}{3}E^1 & 0 & \frac{2}{5}E^2\\
\frac{2}{3}E^1 & 0 & \frac{2}{3}E^0+\frac{4}{15}E^2 & 0 & \frac{4}{15}E^1 & 0\\
0 & \frac{2}{3}E^1 & 0 & \frac{2}{3}E^0+\frac{4}{15}E^2 & 0 & \frac{4}{15}E^1\\
\frac{2}{5}E^2 & 0 & \frac{4}{15}E^1 & 0 & \frac{2}{5}E^0+\frac{4}{35}E^2 & 0\\
0 & \frac{2}{5}E^2 & 0 & \frac{4}{15}E^1 & 0 & \frac{2}{5}E^0+\frac{4}{35}E^2
\end{vmatrix}, \tag{58}$$

$$\mathbf{L}=\begin{vmatrix}
L_{11}^{00} & L_{13}^{00} & 0 & L_{13}^{01} & 0 & 0\\
L_{31}^{00} & L_{33}^{00} & L_{31}^{01} & L_{33}^{01} & 0 & 0\\
L_{11}^{10} & 0 & L_{11}^{11} & L_{13}^{11} & 0 & L_{13}^{12}\\
L_{31}^{10} & L_{33}^{10} & L_{31}^{11} & L_{33}^{11} & 0 & L_{33}^{12}\\
0 & 0 & 0 & L_{13}^{21} & L_{11}^{22} & L_{13}^{22}\\
0 & 0 & L_{31}^{21} & L_{33}^{21} & L_{31}^{22} & L_{33}^{22}
\end{vmatrix},\mathbf{u}=\begin{vmatrix} u_1^0\\ u_3^0\\ u_1^1\\ u_3^1\\ u_1^2\\ u_3^2\end{vmatrix},\mathbf{f}=\begin{vmatrix} f_1^0\\ f_3^0\\ f_1^1\\ f_3^1\\ f_1^2\\ f_3^2\end{vmatrix}. \tag{59}$$

Here again most of the operators L_{ij}^{nm} are differential operators and they have the form

$$L_{11}^{00}u_1^0=(\lambda+2\mu)\frac{\partial^2 u_1^0}{\partial x_1^2}\ ,\ L_{13}^{00}u_3^0=\frac{\lambda}{R}\frac{\partial u_3^0}{\partial x_1},L_{11}^{01}u_1^1=0,\ L_{13}^{01}u_3^1=\frac{\lambda}{h}\frac{\partial u_3^1}{\partial x_1},L_{11}^{02}u_1^2=0,\ L_{12}^{02}u_2^2=0,$$

$$L_{31}^{00}u_1^0=-\frac{\lambda}{R}\frac{\partial u_1^0}{\partial x_1},\ L_{33}^{00}u_3^0=\mu\frac{\partial^2 u_2^0}{\partial x_1^2}-(\lambda+2\mu)\frac{u_3^0}{R},\ L_{31}^{01}u_1^1=\frac{\mu}{h}\frac{\partial u_1^1}{\partial x_1},\ L_{33}^{01}u_3^1=-\frac{\lambda}{Rh}u_3^1,\ L_{31}^{02}u_1^2=0,$$

$$L_{33}^{02}u_3^2=0,$$

$$L_{11}^{10}u_1^0=0\ ,\ L_{13}^{10}u_3^0=-\frac{3\mu}{h}\frac{\partial u_2^0}{\partial x_1},\ L_{11}^{11}u_1^1=(\lambda+2\mu)\frac{\partial^2 u_1^1}{\partial x_1^2}-\frac{3\mu}{h^2}u_1^1,\ L_{13}^{11}u_3^1=\frac{\lambda}{R}\frac{\partial u_3^1}{\partial x_1}+\mu k_1\frac{\partial u_2^1}{\partial x_1},$$

$$L_{11}^{12}u_1^2 = 0\,,\; L_{13}^{12}u_3^2 = \frac{3\lambda}{h}\frac{\partial u_3^2}{\partial x_1}, \tag{60}$$

$$L_{31}^{10}u_1^0 = -\frac{3\lambda}{h}\frac{\partial u_1^0}{\partial x_1}, L_{33}^{10}u_3^0 = -\frac{3\lambda}{Rh}u_3^0, L_{31}^{11}u_1^1 = -\frac{\lambda}{R}\frac{\partial u_1^1}{\partial x_1},$$

$$L_{33}^{11}u_3^1 = \mu\frac{\partial^2 u_2^1}{\partial x_1^2} - (\lambda + 2\mu)\left(\frac{1}{R^2} + \frac{3}{h^2}\right)u_3^1, L_{31}^{12}u_1^2 = 0\,,\; L_{33}^{12}u_3^2 = -\frac{3\lambda}{Rh}u_3^2,$$

$$L_{11}^{20}u_1^0 = 0\,,\; L_{13}^{20}u_3^0 = 0\,, L_{11}^{21}u_1^1 = 0\,, L_{33}^{21}u_3^1 = -\frac{5\mu}{h}\frac{\partial u_3^1}{\partial x_1}\; L_{11}^{22}u_1^2 = (\lambda + 2\mu)\frac{\partial^2 u_1^2}{\partial x_1^2} - \frac{15\mu}{h^2}u_1^2,\; L_{13}^{22}u_3^2 = \frac{\lambda}{R}\frac{\partial u_3^2}{\partial x_1},$$

$$L_{21}^{20}u_1^0 = 0\,,\; L_{33}^{20}u_3^0 = 0\,, L_{12}^{21}u_1^1 = -\frac{5\lambda}{h}\frac{\partial u_1^1}{\partial x_1}, L_{33}^{21}u_3^1 = -\frac{5\lambda}{Rh^2}u_3^1,\; L_{21}^{22}u_1^2 = -\frac{\lambda}{R}\frac{\partial u_1^2}{\partial x_1},$$

$$L_{22}^{22}u_2^2 = \mu\frac{\partial^2 u_2^2}{\partial x_1^2} - (\lambda + 2\mu)\left(\frac{1}{R^2} + \frac{15}{h^2}\right)u_3^2.$$

Components of strain and stress tensors can be calculated using relations

$$\varepsilon_{11}^0 = \frac{1}{A_1}\frac{\partial u_1^0}{\partial x_1} + k_1 u_2^0, \varepsilon_{12}^0 = \frac{1}{A_1}\frac{\partial u_2^0}{\partial x_1} - k_1 u_1^0 + \frac{1}{h}u_1^1, \varepsilon_{22}^0 = \frac{1}{h}u_2^1.$$

$$\varepsilon_{11}^1 = \frac{1}{A_1}\frac{\partial u_1^1}{\partial x_1} + k_1 u_2^1, \varepsilon_{12}^1 = \frac{1}{A_1}\frac{\partial u_2^1}{\partial x_1} - k_1 u_1^1 + \frac{3}{h}u_1^2(x_1), \varepsilon_{22}^1 = \frac{3}{h}u_2^2(x_1) \tag{61}$$

$$\varepsilon_{11}^2 = \frac{1}{A_1}\frac{\partial u_1^2}{\partial x_1} + k_1 u_2^2, \varepsilon_{12}^2 = \frac{1}{A_1}\frac{\partial u_2^2}{\partial x_1} - k_1 u_1^0, \varepsilon_{22}^2 = 0.$$

$$\sigma_{ij}^0 = c_{ijkl}^0\left(2E^0\varepsilon_{kl}^0 + \frac{2}{3}E^1\varepsilon_{kl}^1 + \frac{2}{5}E^2\varepsilon_{kl}^2\right),$$

$$\sigma_{ij}^1 = c_{ijkl}^0\left(\frac{2}{3}E^1\varepsilon_{kl}^0 + \left(\frac{2}{3}E^0 + \frac{4}{15}E^2\right)\varepsilon_{kl}^1 + \frac{4}{15}E^1\varepsilon_{kl}^2\right), \tag{62}$$

$$\sigma_{ij}^2 = c_{ijkl}^0\left(\frac{2}{5}E^2\varepsilon_{kl}^0 + \frac{4}{15}E^1\varepsilon_{kl}^1 + \left(\frac{2}{5}E^0 + \frac{4}{35}E^2\right)\varepsilon_{kl}^2\right).$$

and (57).

7 RESULTS AND DISCUSSIONS

We consider here the case of relatively thick axisymmetric cylindrical shells. Therefore we will keep three members in polynomial expansion (57). In this case we will get the second order approximation equations for functionally graded shells. For this case system of equations for displacements has the form as (34), and corresponding matrixes and vector have the form (59), which together with corresponding boundary conditions can be used for the stress-strain calculation for the second approximation shell theory.

Material properties of an FGM are the functions of volume fractions and they are managed by a volume fraction. When the shell is considered to consist of two materials with Young's modulus E_1 and E_2 respectively, the effective Young's modulus $E(x_3)$ given by the following power-law expression

$$E(x_3) = (E_2 - E_1)\left(\frac{x_3 + h}{2h}\right)^n + E_1 \;\; (n \geq 0). \tag{63}$$

Substituting function (63) into equation (29) we obtain expressions for the Legendre polynomials coefficients for the effective Young's modulus

$$E^1 = \frac{(E_2 + E_1)n}{1+n},\; E^2 = \frac{(E_2 - E_1)nh}{2+3n+n^2},\; E^3 = -\frac{5(E_2 - E_1)(n-1)nh^2}{(1+n)(2+n)(3+n)}, \quad E^2 = \frac{(E_2 - E_1)nh}{2+3n+n^2}. \tag{64}$$

For simplicity in this study we consider dimensionless coordinates $\xi_1 = x_1/L$ and $\xi_3 = x_3/h$ have been introduced. Calculations have been done for Young's modulus equal to $E_1 = 1$ Pa and $E_1/E_2 = 2$ and for Poison ratio $\nu = 0.3$ respectively, other parameters are $R = 0.25L$, $h = 0.25R$ and $n = 0.2$. Numerical calculations have been done using commercial software COMSOL Multiphysics and MATLAB. Results of calculations are presented on Fig. 1–2.

Fig. 1 shows the Legendre polynomials coefficients for the displacements distribution versus the normalized length for the second approximation theory. These coefficients are FEM solutions of the systems of differential equations (34) with matrix operators (58)-(59).

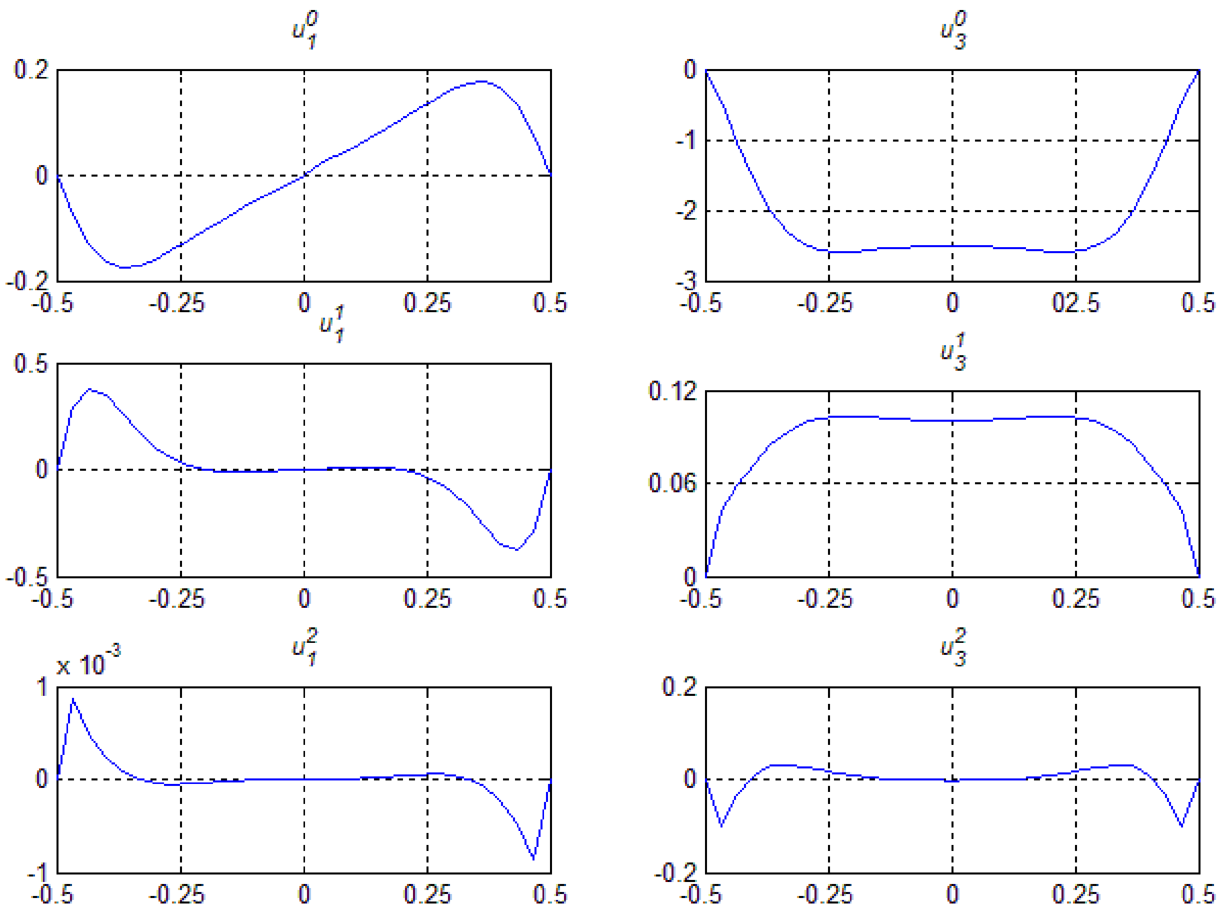

Fig.1. Coefficients of Legendre's polynomials for the displacements for second order approximation.

Fig. 2 shows displacements and stresses distribution versus normalized length and thickness for second approximation theory calculated using equations (62) ans (57).

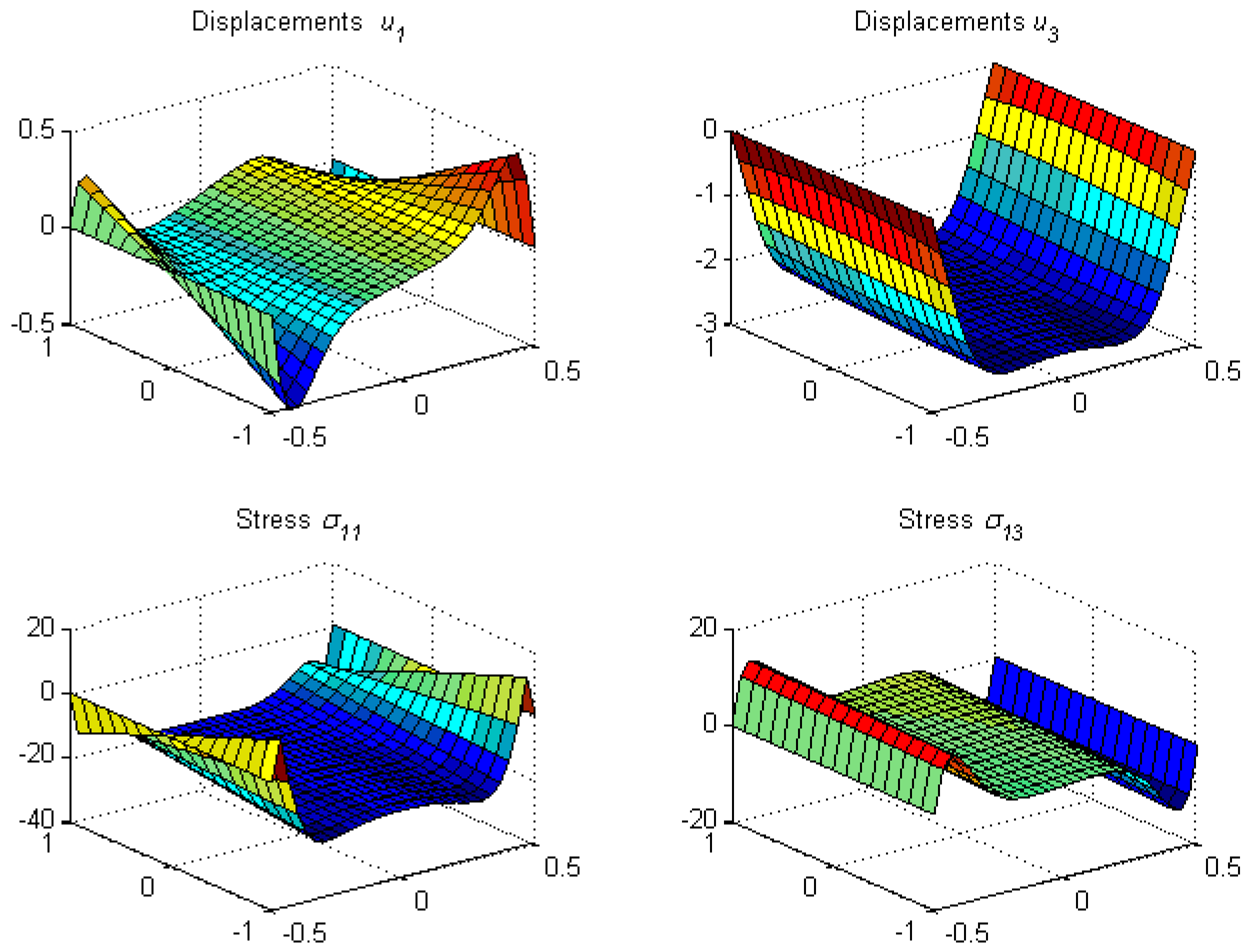

Fig.2. Displacements and stresses versus the normalized length and thickness for second order approximation.

8 CONCLUSION

In this paper a high order theory for FG shells has been developed. The proposed approach is based on the expansion of the axisymmetric equations of elasticity for FGMs into Fourier series in terms of Legendre's polynomials. Starting from the axisymmetric equations of elasticity, the stress and strain tensors, the displacement, traction and body force vectors and the material parameters of FGMs have been expanded into Fourier series in terms of Legendre's polynomials in the thickness coordinate. Thereby all equations of elasticity including Hooke's law have been transformed to the corresponding equations for the series expansion coefficients. The system of differential equations in terms of the displacements and the boundary conditions for the expansion coefficients has been obtained. The first and second order approximations of the exact shell theory have been considered in more details. All necessary equations and their expansion coefficients have been derived explicitly and the corresponding boundary-value problems have been formulated. For the numerical solution of the formulated problems the FEM implemented in the commercial software COMSOL Multiphysics and MATLAB have been used. For the validation of the proposed approximate shell theory a comparison with the results obtained using the exact equations of elasticity has been made.

The influence of the material graduation parameters on the stress-strain state of the FG cylindrical shell has been studied.

Acknowledgment

The work presented in this paper was supported by the Brazilian state foundation FAPERJ, Project E-26/110.011/2013, which is gratefully acknowledged.

REFERENCES

[1] *Altenbach H. Eremeev V. A.* Shell Like Structures. Non Classical Theories and Applications. - Springer, New York, 2011. – 761 p.

[2] *Altenbach H., Eremeyev V. A.* On the shell theory on the nanoscale with surface stresses // International Journal of Engineering Science, 2011, 49, P. 1294–1301.

[3] *Arshad S.H, Naeem M.N, Sultana N., Iqbal Z., Shah A.G.* Effects of exponential volume fraction law on the natural frequencies of FGM cylindrical shells under various boundary conditions // Archive of *Applied* Mechanics – 2011, 81, P. 999-1016.

[4] *Brischetto S, Carrera E.* Advanced mixed theories for bending analysis of functionally graded plates // Computers & Structures - 2010, 88. P. 1474-1483.

[5] *Chung Y.L., Chi S.H.* Mechanical behavior of functionally graded material plates under transverse load - Part I: Analysis // International Journal of Solids and Structures 2006, 43, P. 3657-3674.

[6] *Chung Y.L., Chi S.H.* Mechanical behavior of functionally graded material plates under transverse load - Part II: Numerical results // International Journal of Solids and Structures 2006, 43 P. 3675-3691.

[7] *Duan H. L., Wang J., Karihaloo B. L.* Theory of elasticity at the nanoscale // Advances in Applied Mechanics, vol. 42, pp. 1–68, 2009.

[8] *Gulyaev V.I., Bazhenov, V. A. , Lizunov, P. P.* The Nonclassical Theory of Shells and Its Application to the Solution of Engineering Problems. - L'vov: Vyshcha Shkola, 1978.- 190 p.

[9] *Ebrahimi F, Sepiani H.* Transverse shear and rotary inertia effects on the stability analysis of functionally graded shells under combined static and periodic axial loadings // Journal of Mechanical Science and Technology. 2010, 24, P. 2359-2366.

[10] *Ferreira A.J.M., Batra R.C., Roque C.M.C., Qian L.F., Martins P.A.L.S.* Static analysis of functionally graded plates using third-order shear deformation theory and a meshless method // Composite Structures. 2005, 69, P. 449-457.

[11] *Jalili N.* Piezoelectric-Based Vibration Control. From Macro to Micro-Nano Scale Systems.- Springer, 2010. – 519 p.

[12] *Kantor B.Ya., Zozulya V.V.* Connected problem on contact plate with rigid body though the heat-conducting layer // Doklady Akademii Nauk Ukrainskoy SSR.1988, 4, P.31-33.

[13] *Khoma I. Y.* Generalized Theory of Anisotropic Shells. Kiev: Naukova dumka, 1987. -172 p.

[14] *Kil'chevskii N.A.* Fundamentals of the Analytical Mechanics of Shells. - Kiev: Publisher House of ANUkrSSR, 1963. -355 p.

[15] *Lekhnitskii S.* Anisotropic Plates. - Routledge, 1968. – 534 p.

[16] *Marchuk O.V., Ilchenko Ya.L., Gniedash S.V.* To analysis of stress-strain state of thick cylindrical shells // International Applied Mechanics, 2011, 47(4), P. 119-126.

[17] *Matsunaga H.* Stress analysis of functionally graded plates subjected to thermal and mechanical loadings // Composite Structures. 2009, 87, P. 344-357.

[18] *Mindlin K. D.* An Introduction to the Mathematical Theory of Vibrations of Elastic Plates. - World Scientific, 2006. – 212 p.

[19] *Naghdi P. M.* The Theory of Plates and Shells // S. Fllgge's Handhuch der Physik, VIa/ 2 , C. Truesdell, ed., *Springer*-Verlag, *Berlin*, 1972, P. 425- 640.
[20] *Noor A. K.* Bibliography of monographs and surveys on shells // Applied Mechanics Reviews, 1990, 43(9), P. 223–234.
[21] *Novozhilov V. V.* Thin Shell Theory. Second ed.- Springer, New York, 1971. – 438 p.
[22] *Pelekh B.L.* The generalized theory of shells. L'vov: L'vov University Press, 1978. – 159 p.
[23] *Pelekh B.L., Lazko V.A.* Laminated anisotropic plates and shells with stress concentrators. - Kiev: Naukova dumka, 1982. – 296 p.
[24] *Pelekh B.L., Suhorolskiy M.A.* Contact problems of the theory of elastic anisotropic shells. - Kiev: Naukova dumka. 1980. – 216 p.
[25] *Pietraszkiewicz W.* Addendum to: bibliography of monographs and surveys on shells // Applied Mechanics Reviews, 1992, 45(6), P. 269.
[26] *Reddy J.N.* Analysis of functionally graded plates // International Journal for Numerical Methods in Engineering . 2000, 47, P. 663-684.
[27] *Reddy J.N., Praveen G.N.* Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates // International Journal of Solids and Structures 1998, 35, P. 4457-4476.
[28] *Reddy J. N.* Mechanics of laminated composite plates and shells: theory and analysis. Second ed. - CRC Press LLC, 2004. – 855 p.
[29] *Shah AG, Mahmood T, Naeem MN.* Vibrations of FGM thin cylindrical shells with exponential volume fraction law // Applied Mathematics and Mechanics (English Edition). 2009. 30. P. 607-615.
[30] *Shen H-S.* Functionally graded materials : nonlinear analysis of plates and shells. - CRC Press, Taylor & Francis Group; 2009.
[31] *Shen Y.P., Wu X.H., Chen C.Q., Tian X.G.* A high order theory for functionally graded piezoelectric shells // International Journal of Solids and Structures 2002, 39, P. 5325-5344.
[32] *Shevchenko V.P., Dovbnya E.N., Yartemik V.V.* Shell of arbitrary curvature with system of cracks of different type and geometry // International Applied Mechanics, 2011, 47(4), P. 89-110.
[33] *Shiota I, Miyamoto, Y.* Functionally Graded Materials 1996. In: Prosiding of 4th International Symposium on Functionally Graded Materials. - Tokyo, Japan. : Elsevier: 1997. – 803 p.
[34] *Simsek M., Reddy J.N.* Bending and vibration of functionally graded microbeams using a new higher order beam theory and the modified couple stress theory // International Journal of Engineering Science, 2013, 64, P. 37-53.
[35] *Suresh S, Mortensen A.* Fundamentals of functionally graded materials. In: Processing and Thermomechanical Behavior of Graded Metals and Metal-Ceramic Composites. - London.: IOM Communications Ltd.: 1998. – 165 p.
[36] *Timoshenko S., Woinowsky-Krieger S.* Theory of Plates and Shells. Second ed. - McGraw-Hill, NewYork, 1959.- 580 p.
[37] *Vekua I. N.* Shell Theory, General Methods of Construction. - Pitman Advanced Pub. Program, Boston, 1986. – 302 p.
[38] *Wozniak C., Rychlewska J., Wierzbicki E.* Modelling and analysis of functionally graded laminated shells // Shell Structures: Theory and Applications. 2005, P. 187-190.
[39] *Xiao J.R., Gilhooley .DF., Batra R.C., McCarthy M.A., Gillespie J.W.* Analysis of thick functionally graded plates by using higher-order shear and normal deformable plate theory and MLPG method with radial basis functions // Composite Structures. 2007, P. 80:539-5352.
[40] *Zhang C, Gao X.W., Sladek J., Sladek V.* Fracture analysis of functionally graded materials by a BEM // Composites Science and Technology 2008, 68, P. 1209-1215.

[41] *Zozulya V.V.* Contact cylindrical shell with a rigid body though the heat-conducting layer // Doklady Akademii Nauk Ukrainskoy SSR. 1989, 10, P. 48-51.

[42] *Zozulya V.V.* The combines problem of thermoelastic contact between two plates though a heat conducting layer // Journal of Applied Mathematics and Mechanics. 1989,53(5), P. 622-627.

[43] *Zozulya V.V.* Contact cylindrical shell with a rigid body though the heat-conducting layer in transitional temperature field // Mechanics of Solids, 1991, 2, P. 160-165.

[44] *Zozulya V.V.* Nonperfect contact of laminated shells with considering debonding between laminas in temperature field // Theoretical and Applied mechanics 2006,42, P.92-97.

[45] *Zozulya VV.* Laminated shells with debonding between laminas in temperature field // International Applied Mechanics, 2006, 42(7), P. 842-848.

[46] *Zozulya V.V.* Mathematical Modeling of Pencil-Thin Nuclear Fuel Rods. In: Gupta A., ed. Structural Mechanics in Reactor Technology. - Toronto, Canada. 2007. p. C04-C12.

[47] *Zozulya V. V.* Contact of a shell and rigid body though the heat-conducting layer temperature field // International Journal of Mathematics and Computers in Simulation. 2007, 2, P. 138-45.

[48] *Zozulya V.V.* Contact of the thin-walled structures and rigid body though the heatconducting layer. In: Krope J, Sohrab, S.H., Benra F.-K., eds. Theoretical and Experimental Aspects of Heat and Mass Transfer. Acapulco, Mexico.: WSEAS Press, 2008. p. 145-50.

[49] *Zozulya V. V.* Contact of the thin-walled structures and rigid body though the heat-conducting layer. In: Krope J, Sohrab, S.H., Benra F.-K., eds. Theoretical and Experimental Aspects of Heat and Mass Transfer. - Acapulco, Mexico.: WSEAS Press. - 2008, P. 145-150.

[50] *Zozulya V. V.* Heat transfer between shell and rigid body through the thin heat-conducting layer taking into account mechanical contact. In: Sunden B., Brebbia C.A. eds. Advanced Computational Methods and Experiments in Heat Transfer X.- Southampton: WIT Press,. 2008, 61, P. 81-90.

[51] *Zozulya V. V.* A high order theory for functionally graded shell // World Academy of Science, Engineering and Technology, 2011, 59, P. 779–784.

[52] *Zozulya V.V.* Numerical solution of the Kirchhoff plate bending problem with BEM // ISRN Mechanical Engineering, 2011, Article ID 295904, 14 pages.

[53] *Zozulya V.V.* New high order theory for functionally graded shells // Theoretical and Applied Mechanics. 2012, 4(50), P. 175-183.

[54] *Zozulya V. V.* A high-order theory for functionally graded axially symmetric cylindrical shells // Archive of Applied Mechanics, 2013, 83(3), P. 331–343.

[55] *Zozulya V. V.* A high order theory for linear thermoelastic shells: comparison with classical theories // Journal of Engineering. 2013, Article ID 590480, 19 pages

[56] *Zozulya V.V., Aguilar M.* Thermo-elastic contact and heat transfer between plates and shells through the heat-conducting layer. In: Sunden B., Brebbia C.A. eds. Advanced computational methods in heat transfer VI .- Southampton: WIT Press, 2000, 3. P. 85-94.

[57] *Zozulya V.V., Borodenko Yu.N.* Thermoplastic contact of rigidly fixed shell with a rigid body though the heat-conducting layer // Doklady Akademii Nauk Ukrainskoy SSR. 1991, 7, P.47-53.

[58] *Zozulya V.V. Borodenko, Yu.N.* Connecting problem on contact of cylindrical shells with a rigid body in temperature though the heat-conducting layer // Doklady Akademii Nauk Ukrainskoy SSR. 1992, 4, p. 35-41.

[59] *Zozulya V. V., Zhang Ch.* A high order theory for functionally graded axisymmetric cylindrical shells // International Journal of Mechanical Sciences, 2012, 60(1), P. 12-22.